

Introduction to the Special Issue on the Dick Effect

DURING the course of development of frequency standards based on confined ions, a new question concerning their stability was uncovered: does the noise of the local oscillator influence the long-term performance of the atomic standard. This question came about since trapped ions provided the first opportunity to extend the interaction time of the atoms with the local oscillator derived microwave field to several seconds, or longer. Because the local oscillator in these instruments is tied to the transition in the atom via a frequency locked loop, the influence of noise in the local oscillator on the long-term stability of the atomic standard, during the few seconds that it free runs, remained a question. This concern was prompted by the well-known ideas in the sampling theory which predict aliasing of the noise at high frequencies to frequencies close to the signal, based on the sampling rate.

In 1989, John Dick at JPL investigated the influence of the noise of the local oscillator on the frequency of the mercury ion standard and developed a heuristic picture that was shown to faithfully reproduce the experimental observation. Shortly after this, the advent of laser cooled atomic clocks as realized in the Paris Observatory in France presented the investigators with the reality that the noise of the local oscillator is a concern for all high performance atomic standards. These standards have intrinsically lower noise at short averaging intervals than the available local oscillators they control. Because of this, several investigations were aimed at the further understanding of this phenomenon, which by now had become known as the Dick effect.

In the following four papers the Dick effect is studied from different, and complimentary, perspectives. In the paper by Santarelli *et al.*, the influence of the oscillator noise on the stability of the atomic oscillator is studied by analyzing the quantum mechanical response of a two level atom undergoing a single (Rabi scheme) or multiple (Ramsey scheme) interaction with the applied electromagnetic fields. Audoin *et al.* analyze the frequency control loop of the atomic standard and study various issues, including the influence of the Dick effect on the stability of the standard. Presti *et al.* approach the problem by developing a

model of the phase sampling process with the local oscillator noise as its input, and the response of the standard as its output. Finally, Greenhall's paper presents a careful mathematical derivation of the original formula obtained by Dick, based on the time domain analysis of a local oscillator control loop.

These four papers in essence validate each other's findings, but more importantly, complement each other's approach. They represent a comprehensive and satisfying picture of a problem of major importance to high stability atomic standards currently under development. Their results will be invaluable to the practitioners of the field.

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Special Issue Papers

Properties of an Oscillator Slaved to a Periodically Interrogated Atomic Resonator

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Abstract—In advanced atomic resonators, such as those using a fountain of cold cesium atoms or an ensemble of stored ions, the atomic medium is interrogated periodically, and the control signal of the slaved oscillator is updated at equally spaced time intervals. We analyze the properties of the output frequency of these frequency standards. We establish the equations that describe the time behavior of this frequency. We give the stability condition and the transient response of the frequency feedback loop, the response to systematic frequency changes of the free running oscillator, the frequency stability for given free-running oscillator noise and given optical detection noise, and the limitation of the frequency stability by down-conversion of the intrinsic oscillator frequency noise (Dick effect). We point out that a second integration in the feedback loop may not improve significantly the rejection of slow perturbations, unless a condition relative to the timing of the atom-field interaction is verified.

I. INTRODUCTION

ADVANCED ATOMIC RESONATORS have been developed recently. They are based on microwave transitions observed in mercury or ytterbium ions stored in a radiofrequency trap [1], [2] or in cold cesium atoms launched either in an Earth-bound atomic fountain or in a configuration designed for space operation [3], [4]. They have shown a fractional frequency stability of the order of $1 \cdot 10^{-13} \tau^{-1/2}$, or better, and an accuracy of $2 \cdot 10^{-15}$ in the case of the cesium fountain.

These devices operate sequentially. This is mandatory to observe the atomic transition in the dark, after the atoms or ions have been prepared by optical pumping. Furthermore, the cooling process, when applied, requires a time of atom accumulation before they are launched. Thus, the atoms are not interrogated continuously, as in more traditional atomic resonators, but periodically with dead-time between each atom-field interaction. Furthermore, because the atomic response is available at the end

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of each interrogation process only, the control voltage of the oscillator slaved to the atomic resonance cannot be refreshed otherwise than periodically, at discrete times.

The purpose of this paper is to present a simple but efficient analysis of the salient properties of the frequency control loop of the oscillator that is slaved to such an atomic resonator. This analysis will be applied to the servo-loop associated with the LPTF cesium fountain. We will give briefly indications on the properties of other possible designs. Starting from a time domain description of the dynamical behavior of the feedback loop, we will give the stability condition of the loop and describe the transient response to a step-like perturbation of the oscillator frequency. We will point out possible drawbacks of the considered frequency loop with respect to slow frequency changes of the free running oscillator. We will relate the frequency stability of the slaved oscillator: to the frequency noise that it would display if it were free-running and to the optical detection noise. Finally, we will show how the model accounts for the limitation of the frequency stability of the slaved oscillator by down-conversion of the oscillator intrinsic frequency noise [5], [6].

II. TIME DOMAIN DESCRIPTION OF THE FREQUENCY VARIATION OF THE SLAVED OSCILLATOR

A. Sampling of the Microwave Field by the Atoms

For our purpose, we will assume that a cycle of operation, of duration T_c , begins at time t_k and ends at time t_{k+1} , when a correction is applied to the slaved oscillator. Thanks to the periodicity, the index k can be varied by integer values. As shown in Fig. 1, the interrogation of the atoms starts at t'_k , such that $t'_k - t_k = T_p$ and lasts the time interval T_i . We define $\Delta\omega$ the difference between the angular frequency, ω , of the microwave field and that, ω_o , of the atomic transition. Here, ω is the frequency irrespective of the modulation that is necessary to probe the atomic resonance.

At the discrete times t_k and t_{k+1} , when the control voltage of the slaved oscillator is updated, the relative frequency offset is denoted $\Delta\omega_s(t_k)$ and $\Delta\omega_s(t_{k+1})$, respectively. Although the voltage is held constant between these two instants, the frequency of the oscillator can vary under

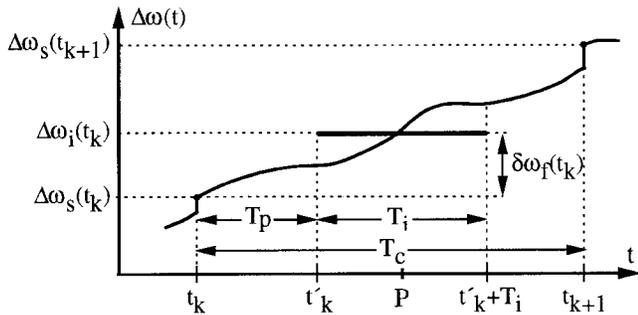


Fig. 1. Frequency change of the microwave field and mean value, $\Delta\omega_i(t_k)$ of this frequency during the interrogation.

the effect of its intrinsic noise or of environmental perturbations. Therefore, for $t_k \leq t < t_{k+1}$, the frequency offset $\Delta\omega_s(t)$ of the microwave field is given by:

$$\Delta\omega_s(t) = \Delta\omega_s(t_k) + \Delta\omega_f(t) - \Delta\omega_f(t_k), \quad (1)$$

where $\Delta\omega_f(t)$ represents the time variation of $\Delta\omega$ when the oscillator is free-running.

The response of the atomic resonator depends on the instantaneous values of $\Delta\omega(t)$ during the atom-microwave interaction. However, we can define an effective frequency offset of the microwave interrogation field during the interaction, $\Delta\omega_i(t_k)$, which is constant and provides the same atomic resonator response as $\Delta\omega(t)$. The definition of $\Delta\omega_i(t_k)$ involves the frequency sensitivity function $g(t)$, first introduced by Dick [5] and Dick *et al.* [6] and considered in more detail in a companion paper [7]. Here, it is sufficient to know that $g(t)$ is a periodic function of time, with period $2T_c$. This function is equal to zero during the dead times when the atom-field interaction does not occur. We have:

$$\Delta\omega_i(t_k) = \Delta\omega_s(t_k) + \delta\omega_f(t_k), \quad (2)$$

with:

$$\delta\omega_f(t_k) = \frac{1}{g_0 T_c} \int_{t_k}^{t_{k+1}} g(t - t_k) [\Delta\omega_f(t) - \Delta\omega_f(t_k)] dt. \quad (3)$$

In this equation, g_0 is the mean value of $g(t)$ during the cycle considered. The quantity $\delta\omega_f(t_k)$ thus represents the average value of the increment of $\Delta\omega_f(t)$ during a cycle, but weighted by $g(t)$.

B. Error Signal

Actually, the microwave field is square-wave frequency modulated, with period $2T_c$. The modulation depth is ω_m . During a period of modulation, we denote:

$$\omega_0 + \Delta\omega_s(t_{k-2}) + \delta\omega_f(t_{k-2}) - (-1)^k \omega_m$$

and

$$\omega_0 + \Delta\omega_s(t_{k-1}) + \delta\omega_f(t_{k-1}) + (-1)^k \omega_m$$

the effective frequency of the microwave field from t_{k-2} to t_{k-1} and t_{k-1} to t_k , respectively.

At time t_k , the difference $\mathcal{D}N(t_k)$ between the detection signals available at times t_k and t_{k-1} is given by:

$$\begin{aligned} \mathcal{D}N(t_k) = & (-1)^k \frac{\partial h}{\partial \omega} N_o [\Delta\omega_s(t_{k-2}) + \delta\omega_f(t_{k-2}) \\ & + \Delta\omega_s(t_{k-1}) + \delta\omega_f(t_{k-1})] - \delta N(t_{k-1}) + \delta N(t_k). \end{aligned} \quad (4)$$

In this equation, N_o and $h(\omega - \omega_o)$ —an even function of $(\omega - \omega_o)$ —represent the peak to valley height of the resonance pattern and its shape, respectively. The derivative is taken at the frequency which was actually applied between t_{k-1} and t_k . It has been assumed that the frequency offsets $\Delta\omega_s$ and $\delta\omega_f$ are very small compared to the atomic resonance line-width. The quantities $\delta N(t_{k-1})$ and $\delta N(t_k)$ are the independent fluctuations of the atomic response, i.e., the detection noise, at the discrete times t_{k-1} and t_k , respectively. The series of values of $\delta N(t_k)$, $\{\delta N(t_k)\}$, is a sequence of independent random variables. We assume that it has the properties of a white noise process, having mean zero and variance $\sigma_{\delta N}^2$.

In the LPTF cesium fountain, the oscillator frequency is corrected at each t_k . To suppress the change of sign occurring between two successive values of $\mathcal{D}N(t_k)$, this quantity is multiplied by $(-1)^k$ and the error signal $\Delta E(t_k)$ is obtained. We have:

$$\begin{aligned} \Delta E(t_k) = & \frac{\partial h}{\partial \omega} N_o [\Delta\omega_s(t_{k-2}) + \delta\omega_f(t_{k-2}) \\ & + \Delta\omega_s(t_{k-1}) + \delta\omega_f(t_{k-1})] + \delta\tilde{N}(t_{k-1}) + \delta\tilde{N}(t_k), \end{aligned} \quad (5)$$

where we have set:

$$\delta\tilde{N}(t_k) = (-1)^k \delta N(t_k). \quad (6)$$

Because $\{\delta N(t_k)\}$ has been assumed a white noise process, the $\delta\tilde{N}(t_k)$ have the same autocovariance coefficients as the $\delta N(t_k)$. Thus, $\{\delta\tilde{N}(t_k)\}$ is also a white noise process with the same variance and power spectral density as $\{\delta N(t_k)\}$.

C. Equation of the Frequency Control Loop

We will consider two different processings of the error signal (others may be contemplated). They mimic the processings accomplished in analog first and second order frequency control loops, respectively.

1. *Case A:* The error signal is added at time t_k to its previously accumulated value $\Delta E'(t_{k-1})$. We thus have:

$$\Delta E'(t_k) = \Delta E'(t_{k-1}) + \Delta E(t_k). \quad (7)$$

This equation is that of a numerical integrator. The signal obtained is applied at time t_k to the varactor of the oscillator. The frequency offset of the slaved oscillator is thus given by:

$$\Delta\omega_s(t_k) = \Delta\omega_f(t_k) + K \Delta E'(t_k), \quad (8)$$

where K is a constant.

The equation that describes the dynamical behavior of the frequency control loop is easily obtained from (5), (7), (8). We have:

$$\begin{aligned} \Delta\omega_s(t_k) - (1 - \beta)\Delta\omega_s(t_{k-1}) + \beta\Delta\omega_s(t_{k-2}) \\ = \Delta\omega_f(t_k) - \Delta\omega_f(t_{k-1}) - \beta[\delta\omega_f(t_{k-1}) + \delta\omega_f(t_{k-2})] \\ + K[\delta\tilde{N}(t_k) + \delta\tilde{N}(t_{k-1})], \quad (9) \end{aligned}$$

with:

$$\beta = -KN_o\partial h/\partial\omega. \quad (10)$$

The latter quantity characterizes the open loop gain.

2. *Case B*: One may wish to improve the filtering of the error signal before it is applied to the control input of the oscillator. The expectation is to improve the long-term behavior of the oscillator when it is perturbed by slow systematic frequency changes. For that purpose, let us consider the additional recursive filter whose input is $\Delta E'(t_k)$ considered previously and whose output is defined by:

$$\begin{aligned} \Delta E''(t_k) = \Delta E''(t_{k-1}) + \frac{T_c}{\tau_2} \Delta E'(t_k) \\ + \frac{\tau_1}{\tau_2} [\Delta E'(t_k) - \Delta E'(t_{k-1})]. \quad (11) \end{aligned}$$

The parameters τ_1 and τ_2 are time constants. For a low enough Fourier frequency, f , its frequency response is $(1 + 2\pi j f \tau_1)/2\pi j f \tau_2$.

Inserting $\Delta E''$ instead of $\Delta E'$ in (8), the equation for the frequency loop becomes:

$$\begin{aligned} \Delta\omega_s(t_k) - (2 - \beta_1 - \beta_2)\Delta\omega_s(t_{k-1}) \\ + (1 + \beta_2)\Delta\omega_s(t_{k-2}) - \beta_1\Delta\omega_s(t_{k-3}) \\ = \Delta\omega_f(t_k) - 2\Delta\omega_f(t_{k-1}) + \Delta\omega_f(t_{k-2}) \\ - (\beta_1 + \beta_2)\delta\omega_f(t_{k-1}) - \beta_2\delta\omega_f(t_{k-2}) + \beta_1\delta\omega_f(t_{k-3}) \\ + (K_1 + K_2)\delta\tilde{N}(t_k) + K_2\delta\tilde{N}(t_{k-1}) - K_1\delta\tilde{N}(t_{k-2}), \quad (12) \end{aligned}$$

with:

$$\beta_1 = \beta\tau_1/\tau_2, \quad \beta_2 = \beta T_c/\tau_2 \quad (13)$$

$$K_1 = K\tau_1/\tau_2, \quad K_2 = K T_c/\tau_2. \quad (14)$$

For a given frequency sensitivity function $g(t)$, (9) or (12) contain all that is known about the properties of the frequency of the slaved oscillator—which is proportional to that of the microwave field—when sampled at times t_k . They are the equations of numerical linear recursive filters of order 2 and 3, respectively. Some of their coefficients are not constants because the quantities $\delta\omega_f(t_k)$ depend on the spectral content of the free-running oscillator, through the filtering process described by (3). It should be noted that the complete description of the behavior of the output frequency of the standard requires (1) in addition to (9) or (12). The first equation gives the instantaneous frequency change between the sampling times t_k and t_{k+1} .

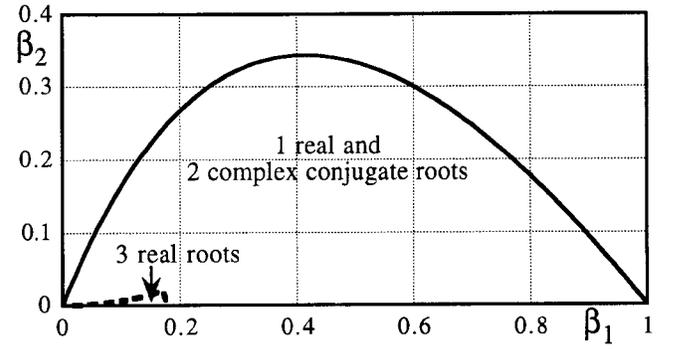


Fig. 2. Stability domain in Case B. It is limited by the curve and the horizontal axis.

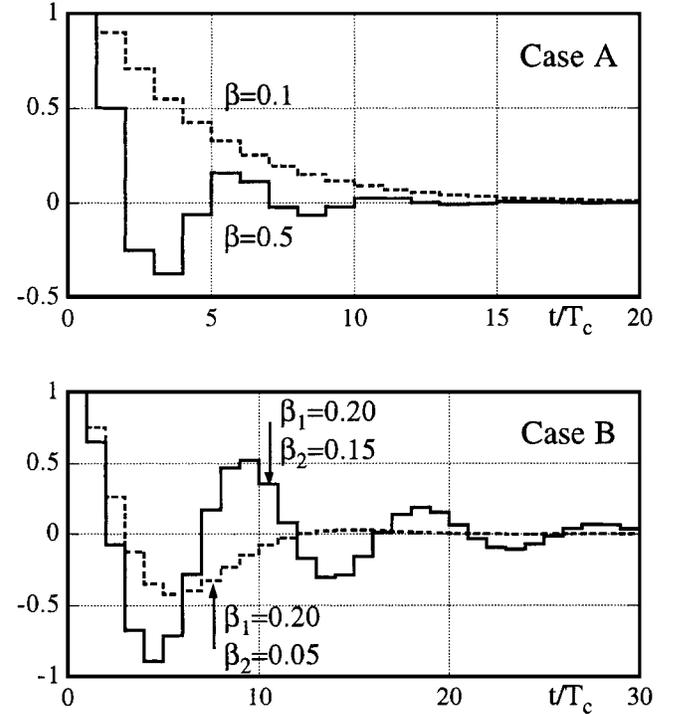


Fig. 3. Response of the frequency feedback loop to a step-function, in Cases A and B.

III. STABILITY CONDITION

Standard techniques of digital filter analysis make it possible to determine the stability condition of the servoloop (see Appendix A). In Case A, it is given by: $0 \leq \beta < 1$. The transient behavior is of the damped type for $0 < \beta \leq 0.172$ and of the oscillatory damped type for $0.172 < \beta < 1$. In Case B, the stability condition is the following:

$$0 < \beta_2 < 2\beta_1(1 - \beta_1)/(1 + \beta_1). \quad (15)$$

Fig. 2 shows the shape of the stability domain in Case B. The transient response is oscillatory damped for most of the allowed values of β_1 and β_2 .

Fig. 3 shows examples of the transient response to a step function applied to the oscillator frequency at the time origin for Cases A and B.

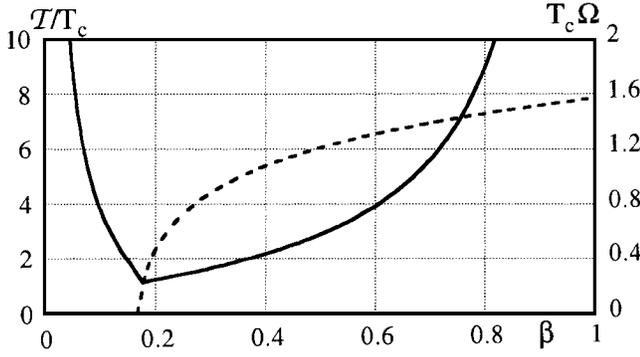


Fig. 4. Time constant (solid line) and pseudo-angular frequency (dashed line) of the transient behavior of the feedback loop in Case A.

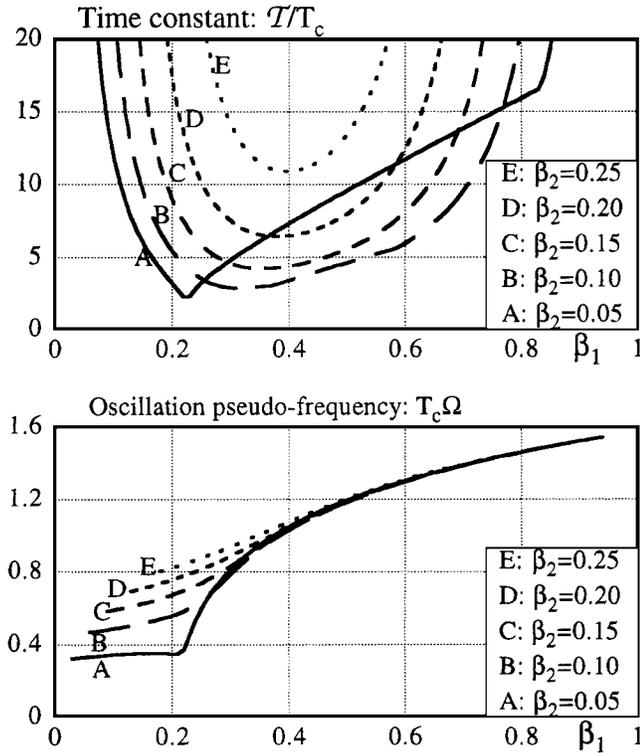


Fig. 5. Time constant and pseudo-angular frequency of the transient behavior of the feedback loop in Case B.

The time constant and the oscillatory pseudo-frequency of the transient response can be defined as shown in Appendix A. Figs. 4 and 5 depict the variation of these parameters versus several values of the loop gains for Cases A and B, respectively.

IV. RESPONSE TO SLOW SYSTEMATIC PERTURBATIONS

A. Response to a Frequency Ramp

The oscillator may show a frequency ramp due to aging of the quartz resonator, for instance. It can be represented by:

$$\Delta\omega_f(t) = \omega_o r t, \quad (16)$$

where r is the slope of the ramp. Then, $\delta\omega_f(t_k)$ is a constant that does not depend on t_k and we set:

$$\delta\omega_f(t_k) = \omega_o r T_1. \quad (17)$$

The time interval T_1 has a specific meaning as we will show immediately. The quantity $g(t)/g_o$ being periodic, with period T_c , we have from (3), (16), and (17):

$$T_1 = \frac{1}{g_o T_c} \int_0^{T_c} \theta g(\theta) d\theta, \quad (18)$$

with $\theta = t - t_k$. Assuming that the frequency sensitivity function would be represented by an even function if the time origin would be set at the middle point between the beginning and the finishing of an atom interrogation process (e.g., at point P in Fig. 1), it can be shown that we have (see Appendix B):

$$T_1 = T_p + T_i/2, \quad (19)$$

where T_p and T_i have been defined in Section 2,A and are shown in Fig. 1. The value of T_1 is thus equal to the time spent from the beginning of the cycle to the middle of the atom-interrogation time interval.

The assumption made holds in the Cs fountain. In the PHARAO experiment [4], it implies that the atom motion is uniform along the axis of the TE_{01n} resonant cavity. This will be the case in space.

Coming back to the loop response, we are interested in the steady state error related to the frequency ramp. We have, from (9), (12), and (17):

$$\frac{\Delta\omega_s(t_k)}{\omega_o} = r \left(\frac{T_c}{2\beta} - T_1 \right) \quad \text{in Case A} \quad (20)$$

$$\frac{\Delta\omega_s(t_k)}{\omega_o} = -r T_1 \quad \text{in Case B.} \quad (21)$$

Because the frequency of the oscillator varies linearly between the frequency corrections, applied at discrete times, the frequency change is rT_c over the duration of a cycle. Fig. 6 shows the saw-tooth shaped instantaneous frequency of the controlled oscillator.

B. Response to a Slow Sinusoidal Frequency Change

The oscillator will likely be perturbed by slow periodic changes of its environmental conditions. This may occur to an orbiting clock, for instance. Let us set:

$$\frac{\Delta\omega_f(t)}{\omega_o} = C_o \sin(2\pi f t). \quad (22)$$

We assume:

$$2\pi f T_c \ll 1, \quad (23)$$

which means that the period of this sinusoidal variation is larger than T_c . Then, between times t_k and t_{k+1} , the frequency offset $\Delta\omega_s(t)$ given by (1) can be written as:

$$\Delta\omega_s(t) = \Delta\omega_s(t_k) + D(t_k)(t - t_k), \quad (24)$$

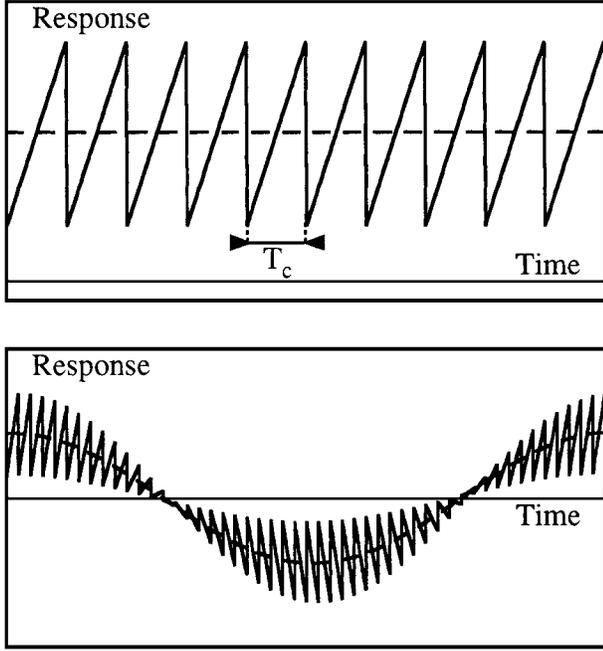


Fig. 6. Top: Response of the feedback loop to a ramp at the input. Bottom: Response to a slow sine-wave at the input. The solid line and the dashed line represent the instantaneous frequency and the mean frequency, respectively.

where $D(t_k)$ is the derivative of $\Delta\omega_f(t)$ at time t_k .

According to the approximation made, $\Delta\omega_s(t)$ varies linearly between t_k and t_{k+1} , with a slope which changes very slowly. The results of Section IV, A can thus be applied and we find, to first order with respect to $2\pi fT_c$:

$$\frac{\Delta\omega_s(t_k)}{\omega_o} = 2\pi fT_c \left(\frac{1}{2\beta} - \frac{T_1}{T_c} \right) C_o \cos(2\pi ft_k) \text{ in Case A} \quad (25)$$

$$\frac{\Delta\omega_s(t_k)}{\omega_o} = -2\pi fT_1 C_o \cos(2\pi ft_k) \text{ in Case B.} \quad (26)$$

The presence of the term T_1/T_c has been verified experimentally in case A [8].

Besides the residual sampled sinusoidal variation, $\Delta\omega_s(t_k)$, of the frequency of the controlled oscillator, a linear frequency change, of amplitude $2\pi fT_c \cos(2\pi ft_k)$, occurs during a cycle. Fig. 6 illustrates the variation of the instantaneous frequency of the controlled oscillator.

C. Practical Consequences

The fast frequency modulation which reflects the frequency change of the oscillator between two corrections has, in fact, a very small amplitude. It will likely be blurred by the random frequency fluctuations due to the optical detection noise.

This does not mean that the effect of this fast modulation can be neglected. Any characterization of the frequency or time stability of the atomic frequency standard involves an averaging of the instantaneous frequency of the

controlled oscillator over a time interval τ . In this paper, we will mainly consider the long-term frequency stability, such that the condition $\tau \gg T_c$ is fulfilled. Consequently, we have to deal with the mean value of the instantaneous frequency. In that case, a term proportional to $T_c/2$, due to the averaging of the saw-tooth-like variation must be added to the right-hand side of (20), (21) and (25), (26). The measurable relative frequency offset, y , is thus the following:

For a ramp at the input:

$$y = r \left(\frac{T_c}{2\beta} - T_1 + \frac{T_c}{2} \right) \text{ in Case A} \quad (27)$$

$$y = r \left(-T_1 + \frac{T_c}{2} \right) \text{ in Case B.} \quad (28)$$

For a sine-wave at the input:

$$y = 2\pi f \left(\frac{T_c}{2\beta} - T_1 + \frac{T_c}{2} \right) C_o \cos(2\pi ft) \text{ in Case A} \quad (29)$$

$$y = 2\pi f \left(-T_1 + \frac{T_c}{2} \right) C_o \cos(2\pi ft) \text{ in Case B.} \quad (30)$$

This frequency offset depends on the loop gain in Case A and, in both cases, on the detail and the duration of the atom-field interaction (through T_i) and its timing (through T_p). The rejection factor of a slow sine-wave applied at the input can be easily derived by comparing (25) and (26) to (22).

It is of interest to note that, in Case B, the steady state error due to a frequency ramp is not equal to zero and the rejection factor of a sine-wave is not of the second order with respect to $2\pi fT_c$. This is a weakness compared to the more traditional situation where the interrogation is performed continuously.

However, this drawback disappears if the following condition is fulfilled:

$$T_1 = T_c/2. \quad (31)$$

It practically means that the atom-field interaction must be centered on the cycle of duration T_c . Then, one retrieves the familiar properties of continuous time analog frequency control loops. The steady state error and the rejection factor, respectively, given by (27) and (29) are inversely proportional to the open loop gain in Case A. In Case B, the steady state error is equal to zero and the rejection factor is proportional to f^2 , as it will be shown in Section V.

V. FREQUENCY INSTABILITY RELATED TO OSCILLATOR NOISE

In this section, we consider the noise spectral components of the free-running oscillator at Fourier frequencies much smaller than $1/T_c$. If $S_y^f(f)$ is the power spectral density of this noise, then the power spectral density, $S_y(f)$, of

$$H'_{s,f} \left(f \ll \frac{1}{T_c} \right) = \frac{\exp(2\pi i f T_c) [\exp(2\pi i f T_c) - 1] - \beta \left(\frac{g^c}{g_o} - 1 + i \frac{g^s}{g_o} \right) [1 + \exp(2\pi i f T_c)]}{\exp(4\pi i f T_c) - (1 + \beta) \exp(2\pi i f T_c) + \beta}, \quad (33)$$

the relative frequency fluctuations of the slaved oscillator is given by:

$$S_y(f) = |H_{s,f}|^2 S_y^f(f). \quad (32)$$

The transfer function $H_{s,f}$ is composed of two parts. The first one is the frequency response $H'_{s,f}$ of the sampled feedback loop. The second one is associated with the averaging of the frequency change of the free-running oscillator occurring between two successive sampling times.

1. *Case A:* We obtain, from (9):

See Equation (33) at top of page.

with:

$$\left(\frac{g^s}{g^c} \right) = \frac{1}{T_c} \int_0^{T_c} g(\theta) \begin{pmatrix} \sin(2\pi f \theta) \\ \cos(2\pi f \theta) \end{pmatrix} d\theta. \quad (34)$$

The coefficients g^s and g^c depend on the input frequency f . Under assumption (23), we have, to first order:

$$H'_{s,f} = 2\pi i f T_c \left(\frac{1}{2\beta} - \frac{T_1}{T_c} \right), \quad (35)$$

where T_1 is given by (18).

As stated previously, the long-term stability involves the mean value, $m(t_k)$, of the fast frequency modulation occurring between times t_k and t_{k+1} . We have:

$$m(t_k) = \frac{1}{T_c} \int_{t_k}^{t_{k+1}} [\Delta\omega_f(t) - \Delta\omega_f(t_k)] dt. \quad (36)$$

The associated frequency response, H_m , expanded to second order is:

$$H_m = \pi i f T_c - \frac{2}{3} (\pi f T_c)^2. \quad (37)$$

At low Fourier frequencies, the effective frequency response is $H_{s,f} = H'_{s,f} + H_m$. To first order it is given by:

$$H_{s,f} = 2\pi i f T_c \left(\frac{1}{2\beta} + \frac{1}{2} - \frac{T_1}{T_c} \right). \quad (38)$$

This result is, of course, in agreement with (29).

Assuming condition (23), the flicker noise of frequency of the free-running oscillator is converted into flicker noise of phase, whose Allan variance varies as τ^{-2} . In practice, the rejection of the flicker noise of frequency by the frequency control loop is sufficient, for $2\pi f T_c \ll 1$, when good oscillators, including state of the art VCXO are used.

2. *Case B:* Following the same lines as previously, the second order expanded frequency response is given in that case by:

$$H_{s,f} = 2\pi i f T_c \left(\frac{1}{2} - \frac{T_1}{T_c} \right) - (2\pi f T_c)^2 \left(\frac{1}{2\beta_2} + \frac{1}{6} - \frac{1}{2} \frac{T_2^2}{T_c^2} \right), \quad (39)$$

where T_2 is related to the second moment of $g(\theta)$ by:

$$T_2^2 = \frac{1}{g_o T_c} \int_0^{T_c} \theta^2 g(\theta) d\theta. \quad (40)$$

When condition (31) is satisfied, the attenuation of the low frequency noise of the oscillator is much improved, as well as its slow systematic frequency changes.

VI. FREQUENCY INSTABILITY RELATED TO DETECTION NOISE

Actually, the low frequency noise of the free-running oscillator can be neglected, for $2\pi f T_c \ll 1$. Then, the slaved oscillator is perturbed by the detection noise $\delta\tilde{N}$ occurring at discrete times t_k and t_{k+1} , defined by (6). Because the time series $\{\delta\tilde{N}(t_k)\}$ represents a white noise process with variance $\sigma_{\delta\tilde{N}}^2$, as stated in Section II,B, the one-sided power spectral density of the detection noise is given by:

$$S_{\delta\tilde{N}}(f) = 2T_c \sigma_{\delta\tilde{N}}^2, \quad \text{for } 0 \leq f \leq 1/2T_c. \quad (41)$$

It is equal to zero for $f > 1/2T_c$.

The one-sided power spectral density of the relative frequency fluctuations of the controlled oscillator, due to detection noise, is given by:

$$S_y(f) = \frac{1}{\omega_o^2} |H_{s,n}|^2 S_{\delta\tilde{N}}(f), \quad (42)$$

where $H_{s,n}$ is the frequency response to the input noise $\delta\tilde{N}$.

1. *Case A:* We have, from (9)

$$\frac{|H_{s,n}|^2}{K^2} = \frac{(1 + \cos(2\pi f T_c))}{1 - \beta + \beta^2 - (1 - \beta^2) \cos(2\pi f T_c) + \beta \cos(4\pi f T_c)}. \quad (43)$$

Fig. 7 shows the variation of $\beta^2 |H_{s,n}|^2 / K^2$ versus $f T_c \leq 1/2$, for several values of β . For low enough Fourier frequencies, verifying (23), the power transfer function tends to a limit given by:

$$|H_{s,n}(f \rightarrow 0)|^2 = K^2 / \beta^2. \quad (44)$$

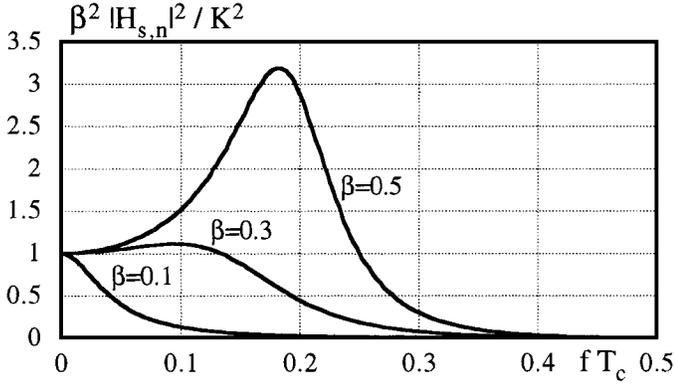


Fig. 7. Power transfer function of the feedback loop relative to the detection noise in Case A.

With (10), (41), and (42), we obtain:

$$S_y(f \rightarrow 0) = \frac{2T_c}{\omega_o^2(\partial h/\partial \omega)^2} \frac{\sigma_{\delta N}^2}{N_o^2}. \quad (45)$$

In the special case where the atom-field interaction takes place according to the Ramsey method [9], we have:

$$h(\omega - \omega_o) = [1 + \cos(\omega - \omega_o)T]/2, \quad (46)$$

where it is assumed that the transit time T between the two oscillatory fields is much larger than the time spent in each of them. Usually, the modulation depth is equal to the half-width at half-maximum. We thus have $|\partial h/\partial \omega| = T/2$. Then, the quality factor Q_{at} of the atomic resonance being equal to $\omega_o T/\pi$, (45) becomes:

$$S_y(f \rightarrow 0) = \frac{8T_c}{\pi^2 Q_{at}^2} \frac{\sigma_{\delta N}^2}{N_o^2}. \quad (47)$$

The related Allan variance is the following, for $\tau \gg T_c$:

$$\sigma_y^2(\tau) = \frac{4}{\pi^2 Q_{at}^2} \frac{\sigma_{\delta N}^2}{N_o^2} \frac{T_c}{\tau}. \quad (48)$$

In the servo-loop, the quantity:

$$\nabla N(t_k) = \delta N(t_k) - \delta N(t_{k-1}), \quad (49)$$

available after the atomic line has been probed on both sides, is more easily accessible to measurement than $\delta N(t_k)$. It is thus useful to express the result of interest using $\sigma_{\nabla N}^2$ instead $\sigma_{\delta N}^2$. We have, obviously:

$$\sigma_{\nabla N}^2 = 2\sigma_{\delta N}^2, \quad (50)$$

and the Allan variance becomes:

$$\sigma_y^2(\tau) = \frac{2}{\pi^2 Q_{at}^2} \frac{\sigma_{\nabla N}^2}{N_o^2} \frac{T_c}{\tau}. \quad (51)$$

Strictly speaking, the measurement of the mean relative frequency averaged over the sampling time τ , \bar{y} , which is implied in the definition of the Allan variance, should start at a time t_k and last $\tau = \ell T_c$, where ℓ is an integer. However, because we can neglect the oscillator intrinsic noise,

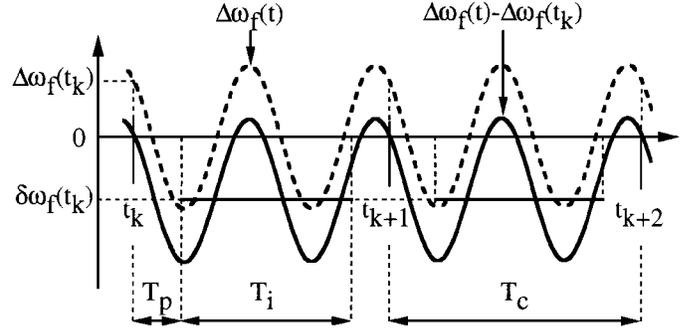


Fig. 8. Illustration of the stroboscopic effect: when the input frequency is equal to a multiple of $1/T_c$, the averaged value $\delta\omega_f(t_k)$ is the same during each cycle.

the output frequency is a constant during any time interval $[t_k, t_{k+1}]$. Therefore, for τ large enough compared to T_c , the error made in relaxing the aforementioned constraints can be neglected.

Assuming, for instance, $Q_{at} = 10^{10}$, $T_c = 1$ s, and $\sigma_{\nabla N}/N_o = 2.2 \times 10^{-3}$, we have $\sigma_y(\tau) = 1.0 \times 10^{-13} \tau^{-1/2}$.

2. *Case B*: From (10) and assuming $fT_c \ll 1$, we have:

$$|H_{s,n}(f \rightarrow 0)|^2 = K_2^2/\beta_2^2 = K^2/\beta^2. \quad (52)$$

This power transfer function is identical to that obtained in Case A. Therefore, the equation for the detection noise limited long-term frequency stability of the controlled oscillator is the same as in Case A. More generally, it can be shown that (45) does not depend on the details of the processing of the error signal.

It should be noted that (48) or (51), which has been derived rigorously, differs by a numerical factor from other ones previously published [3], [10].

VII. FREQUENCY STABILITY LIMITATION BY DOWN CONVERSION OF THE OSCILLATOR NOISE

A. Physical Origin of the Effect

Spectral components of the frequency noise of the free-running oscillator whose frequencies are larger than the feedback-loop cut-off frequency are not filtered out. This is the case, in particular, for the noise components whose frequencies are equal to $1/T_c$ or to a multiple of this value. Fig. 8 shows an example of the related change of $\Delta\omega_f(t) - \Delta\omega_f(t_k)$, assuming a pure sine-wave. In that case the atom-field interaction occurring periodically, during the time interval T_i , is synchronized with the perturbation under consideration. Therefore, the atoms experience the same frequency departure $\delta\omega_f(t_k)$, given by (3), from cycle to cycle. We thus have the equivalent of a stroboscopic effect. This frequency departure causes a permanent frequency offset of the slaved oscillator, which is specified in the following.

Actually, the disturbing signal occupies a finite bandwidth. Consequently, the value of $\Delta\omega_f(t_k)$ and of $\delta\omega_f(t_k)$

fluctuates randomly, but slowly, from one cycle to the following one. This is the origin of the additional low frequency noise of the slaved oscillator, which has been first pointed out and calculated by Dick [5] and Dick *et al.* [6]. Here, we will derive differently the related spurious frequency instability.

B. Frequency Stability Limitation

We will consider the limitation of the long-term frequency stability, i.e., for observation times τ much larger than T_c . The frequency measurement process filters the noise observed and its Fourier frequencies larger than $1/\tau$ are much attenuated. It is thus justified to assume that the bandwidth, Δf , of the down-converted noise is of the order of $1/\tau$ and, consequently, much smaller than $1/T_c$.

The low frequency noise in the bandwidth Δf comes from spectral components of the oscillator noise around Fourier frequencies m/T_c . Then, it suffices to consider that part of the oscillator noise that is filtered in a set of spectral windows centered around frequencies m/T_c and having a noise bandwidth $2\Delta f$. The Rice representation [11] of this narrow band limited noise is the following:

$$\Delta\omega_f^F(t) = \sum_{m=1}^{\infty} \left[p_m(t) \sin\left(2\pi m \frac{t-t_k}{T_c}\right) + q_m(t) \cos\left(2\pi m \frac{t-t_k}{T_c}\right) \right], \quad (53)$$

where $p_m(t)$ and $q_m(t)$ are slowly variable random amplitudes. The constant phase, $-2\pi m t_k/T_c$, is introduced for convenience and it does not change the final result. The one-sided power spectral density of $p_m(t)$ and $q_m(t)$ is related to that of $\Delta\omega_f$ around m/T_c by:

$$S_{p_m}(f \leq \Delta f) = S_{q_m}(f \leq \Delta f) = 2S_{\Delta\omega_f}(f = m/T_c). \quad (54)$$

The frequency offset $\delta\omega_f(t_k)$ is calculated by substituting $\Delta\omega_f^F(t)$ for $\Delta\omega_f(t)$ in (3). According to the assumptions made, $p_m(t)$ and $q_m(t)$ vary very little during the time interval T_c and we can take their value at t_k . We obtain:

$$\delta\omega_f(t_k) = -\Delta\omega_f^F(t_k) + \frac{1}{g_o} \sum_{m=1}^{\infty} [g_m^s p_m(t_k) + g_m^c q_m(t_k)], \quad (55)$$

with:

$$\frac{1}{g_o} \begin{pmatrix} g_m^s \\ g_m^c \end{pmatrix} = \frac{1}{T_c} \int_0^{T_c} \frac{g(\theta)}{g_o} \begin{pmatrix} \sin(2\pi m\theta/T_c) \\ \cos(2\pi m\theta/T_c) \end{pmatrix} d\theta, \quad (56)$$

where $g(\theta)/g_o$ is defined during the time interval $[0, T_c]$. Here, the values of g_m^s/g_o and g_m^c/g_o depend on their rank $m \geq 1$, besides the shape of $g(\theta)/g_o$.

Because $\delta\omega_f^F(t_k)$ and $\delta\omega_f(t_k)$ are slowly varying terms, (9) or (12) shows that the slow change of $\Delta\omega_s(t_k)$ is given by:

$$\Delta\omega_s(t_k) = -\delta\omega_f(t_k). \quad (57)$$

The frequency fluctuation of the microwave field, which is proportional to that of the controlled oscillator, is given by (1). Because we are looking for frequency changes whose spectrum is included in the bandwidth $[0, \Delta f]$, the term $\Delta\omega_f^F(t)$ is irrelevant. We thus have, with (57):

$$\Delta\omega(t_k) = -\delta\omega_f(t_k) - \Delta\omega_f^F(t_k). \quad (58)$$

The frequency of the oscillator being hold from one frequency correction to the other and its changes being very small from one cycle to the following one, we can smooth its fluctuations and write from (55) and (58):

$$\Delta\omega(t) = -\frac{1}{g_o} \sum_{m=1}^{\infty} [g_m^s p_m(t) + g_m^c q_m(t)], \quad (59)$$

where $p_m(t)$ and $q_m(t)$ is white noise in the narrow bandwidth considered.

The power spectral density of the relative frequency fluctuations of the controlled oscillator is easily derived from (54) and (59). The related Allan variance is given by:

$$\sigma_{y,\text{lim}}^2(\tau) = \frac{1}{\tau} \sum_{m=1}^{\infty} \left[\left(\frac{g_m^c}{g_o} \right)^2 + \left(\frac{g_m^s}{g_o} \right)^2 \right] S_y^f \left(\frac{m}{T_o} \right), \quad (60)$$

where $S_y^f(m/T_c) = S_{\Delta\omega_f}^f(m/T_c)/\omega_o^2$ is the one-sided power spectral density, at frequencies m/T_c , of the relative frequency fluctuations of the free-running oscillator. This equation specifies, in agreement with results derived in [5], [6], [12], [13], the annoying frequency instability resulting from the down-conversion of the oscillator noise spectral components around frequencies m/T_c , where m is an integer. The size of the effect depends on two factors. The first one is the power spectral density of the oscillator frequency noise at frequencies m/T_c . The second one is related to the details of the atom-field interaction, which determines the frequency sensitivity function $g(t)$. These two properties have been verified experimentally [7]. The level of the frequency stability limitation does not depend on the particular processing of the error signal.

VIII. CONCLUSION

From simple calculations guided by physical insight, we have established the main properties of an oscillator controlled by an atomic resonator operated sequentially. We have given the stability condition of the feedback loop. We have established the condition to be satisfied to improve significantly the rejection of systematic frequency changes of the oscillator when a second numerical integration is included in the control algorithm. We have derived rigorously the expression for the Allan variance of the relative frequency fluctuations associated with the detection noise. Finally, we have confirmed, by a different approach, the equation giving the frequency limitation originating in the down-conversion of the oscillator frequency noise.

The results given apply specifically to the signal processing implemented in the LPTF cesium fountain, where a frequency correction is applied at the end of each atom interrogation cycle and where the error signal is derived from the difference, $N(t_k) - N(t_{k-1})$ between the last two atomic responses. However, other signal processings can be considered. For instance, the oscillator frequency can be corrected every two cycles. A still different method is used in mercury ion frequency standards, where the error signal is obtained from the following grouping: $N(t_k) - 2N(t_{k-1}) + N(t_{k-2})$ [14], [15]. In all the cases that we have considered, the condition (31) must be satisfied to reject efficiently the slow frequency perturbations of the oscillator.

APPENDIX A

A. Stability Condition

The feedback loop is stable if the modulus of the roots of its characteristic equation is smaller than unity. This equation is:

$$Z^2 - (1 - \beta)Z + \beta = 0 \quad (\text{A-1})$$

$$Z^3 - (2 - \beta_1 - \beta_2)Z^2 + (1 + \beta_2)Z - \beta_1 = 0, \quad (\text{A-2})$$

in Cases A and B respectively. We have set $Z = \exp(2\pi i f t)$. The stability condition can be obtained by applying the Jury criterion, for instance [16].

B. Time Constant and Oscillatory Pseudo-frequency of the Transient Response

We define these parameters by comparing the damped or the oscillatory damped transient response of the servo-loop to that of an analog circuit.

First, assuming a real root, ρ , the response contains a term such as:

$$R(t_k) = C\rho^{t_k/T_c}, \quad (\text{A-3})$$

where C is a constant. If the transient response were a continuous time exponentially damped motion, with time constant \mathcal{T} , we would have:

$$R(t_k) = C \exp(-t/\mathcal{T}). \quad (\text{A-4})$$

We extend the definition of a time constant to the discontinuous motion described by (A-3) by setting:

$$\mathcal{T} = -T_c/\ln(|\rho|). \quad (\text{A-5})$$

Second, for a complex root whose real and imaginary parts are a and b , respectively, we have:

$$R(t_k) = C[\rho \exp(i\theta)]^{t_k/T_c}, \quad (\text{A-6})$$

with:

$$\rho = (a^2 + b^2)^{1/2} \quad \text{and} \quad \tan \theta = b/a. \quad (\text{A-7})$$

By analogy with a continuous time oscillatory damped motion, represented by $\exp[i\Omega - \mathcal{T}^{-1}]t$, we introduce the time constant and the pseudo-oscillatory angular frequency of the actual response by:

$$\mathcal{T} = -T_c/\ln(\rho) \quad \text{and} \quad T_c\Omega = \arctan(b/a). \quad (\text{A-8})$$

The quantities \mathcal{T} and Ω provide an approximate but useful characterization of the loop transient response. (A-1) and (A-2) having two or three roots, respectively, we have retained the largest value of the related time constants.

APPENDIX B

Let us consider that part of the frequency sensitivity function which is defined over the time interval $[t_k, t_{k+1}]$, where we set $\theta = t - t_k$. We translate the time origin to point P shown in Fig. 1, such that $\theta = T_p + T_i/2$, and introduce the new variable $\theta' = \theta - T_p - T_i/2$. Because $g(\theta')$ is equal to zero except for $-T_i/2 \leq \theta' \leq T_i/2$, we have:

$$\frac{1}{g_0 T_c} \int_0^{T_c} \theta g(\theta) d\theta = \frac{1}{g_0 T_c} \left[(T_p + T_i/2) \int_{-T_i/2}^{T_i/2} g(\theta') d\theta' + \int_{-T_i/2}^{T_i/2} \theta' g(\theta') d\theta' \right]. \quad (\text{B-1})$$

In the right-hand side of (B-1), the first integral is equal to $g_0 T_c$. If $g(\theta')$ is an even function of θ' , then $\theta' g(\theta')$ is an odd function and the second integral is equal to zero. It follows that the first moment of $g(\theta)$ is such that:

$$\frac{1}{g_0 T_c} \int_0^{T_c} \theta g(\theta) d\theta = T_p + T_i/2. \quad (\text{B-2})$$

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Since 1992, he has designed and realized the first cesium fountain primary frequency standard.

Frequency Stability Degradation of an Oscillator Slaved to a Periodically Interrogated Atomic Resonator

Giorgio Santarelli, Claude Audoin, Ala'a Makdissi, Philippe Laurent, G. John Dick, and André Clairon

Abstract—Atomic frequency standards using trapped ions or cold atoms work intrinsically in a pulsed mode. Theoretically and experimentally, this mode of operation has been shown to lead to a degradation of the frequency stability due to the frequency noise of the interrogation oscillator.

In this paper a physical analysis of this effect has been made by evaluating the response of a two-level atom to the interrogation oscillator phase noise in Ramsey and multi-Rabi interrogation schemes using a standard quantum mechanical approach. This response is then used to calculate the degradation of the frequency stability of a pulsed atomic frequency standard such as an atomic fountain or an ion trap standard. Comparison is made to an experimental evaluation of this effect in the LPTF Cs fountain frequency standard, showing excellent agreement.

I. INTRODUCTION

THE DEVELOPMENT of new passive frequency standards using trapped ions or cold atoms has produced devices with a potential fractional frequency stability of the order of $10^{-13}\tau^{-1/2}$ or better. In these new types of standards, the internal interrogation process is discontinuous and periodic, and the control of the interrogation oscillator also is periodic. The frequency of this oscillator is compared to that of the atomic resonance during a part of duration T_i of the operating cycle only, and its frequency is controlled at the end of each cycle.

In the late 1980s, Dick [1], at the Jet Propulsion Laboratory, derived the atomic response to the oscillator frequency fluctuations using a geometrical approach. Furthermore, it was shown that the oscillator frequency noise at Fourier frequencies, which are close to multiples of $1/T_c$, is down-converted, leading to a degradation of the frequency stability.

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In this paper, we evaluate the atomic response to a variation of the frequency of the interrogation oscillator for different interrogation schemes. This response is used in two companion papers to derive the equation for the frequency stability limitation of the standard [2], [3]. The resulting model is experimentally verified using an atomic fountain standard [4].

II. TIME DEPENDENCE OF THE SENSITIVITY TO OSCILLATOR FREQUENCY FLUCTUATIONS

Let δP be the change of the probability that a transition occurred at the outcome of the atomic interaction with the noisy microwave field of the interrogation oscillator. This change is related to a fluctuation, $\delta\omega(t)$, of the frequency of the interrogating oscillator during the interaction in the following way:

$$\delta P = \frac{1}{2} \int_{\text{int.}} g(t) \delta\omega(t) dt. \quad (1)$$

This equation defines $g(t)$, the sensitivity function to frequency fluctuations of the interrogating field $\delta\omega(t)$. This equation assumes that all the atoms are subjected to the same phase perturbation. The interrogation occurs during an interaction time T_i . The physical meaning of this function can be obtained by calculating the effect of an infinitesimally small phase step $\Delta\phi$ at time t in the oscillator signal, which can be expressed as a frequency variation $\delta\omega(t) = \Delta\phi\delta(t-t')$. This produces a change $\delta P(t, \Delta\phi)$ in the probability that a transition occurred, and $g(t)$ is given by:

$$g(t) = 2 \lim_{\Delta\phi \rightarrow 0} \delta P(t, \Delta\phi) / \Delta\phi. \quad (2)$$

In control system terminology, $g(t)$ is the response of the atomic system to a phase step of the interrogation oscillator, or the impulse response with respect to a frequency change occurring at time t .

A. A Simple Example of $g(t)$

We consider a Ramsey interrogation scheme where the atoms have a resonance frequency ω_0 and experience successively two microwave fields of frequency ω and Rabi frequency b , each for a duration τ_p , separated by a time T .

Assuming $T \gg \tau_p$ and $\Omega_0 = (\omega - \omega_0) \ll b$, the probability that the atomic transition occurred can be written as:

$$P \cong \frac{1}{2} \sin^2 b\tau_p [1 + \cos(\Omega_0 T + \Delta\phi)], \quad (3)$$

where $\Delta\phi$ is a phase step that can take place at any time between the two microwave interactions.

It is worth noting that the interrogating field is stepwise frequency modulated with frequency $1/2T_c$, where T_c is the time for one cycle of operation, to generate the servo error signal required to lock the interrogation oscillator on the atomic transition. Its (angular) frequency is $\omega_0 + \Delta\omega + \omega_m$ or $\omega_0 + \Delta\omega - \omega_m$ according to the half period of modulation considered, where $\Delta\omega \ll b$ is the difference between the oscillator frequency and the resonance frequency ω_0 and ω_m is the modulation depth. If we consider a small phase step $\Delta\phi \ll 1$, applying (3) to the generic k -th cycle, we obtain:

$$P_k \cong \frac{1}{2} \sin^2 b\tau_p [1 + (\cos((\Delta\omega + (-1)^k \omega_m)T) - \sin((\Delta\omega - (-1)^k \omega_m)T)\Delta\phi)]. \quad (4)$$

When the interrogation oscillator is locked to the resonance, $\Delta\omega \approx 0$. Then the variation of the signal due to $\Delta\phi$ becomes:

$$\delta P_k \cong \frac{(-1)^k}{2} \sin^2 b\tau_p \sin(\omega_m T) \Delta\phi \quad (5)$$

and using the relation (2) we obtain

$$g(t) = \begin{cases} (-1)^k \sin^2 b\tau_p \sin \omega_m T & 0 \leq t \leq T, \\ 0 & T \leq t \leq T_c. \end{cases} \quad (6)$$

This function is periodic, with period $2T_c$. As explained later and in two companion papers [2], [3], a meaningful quantity in this process is $g(t)/g_0$ where g_0 is the mean value of $g(t)$ over the cycle time T_c :

$$g_0 = \frac{1}{T_c} \int_0^{T_c} g(t) dt \approx (-1)^k \frac{T}{T_c} \sin^2 b\tau_p \sin \omega_m T. \quad (7)$$

We thus have:

$$g(t)/g_0 = \begin{cases} \frac{T}{T_c} & 0 \leq t \leq T, \\ 0 & T \leq t \leq T_c. \end{cases} \quad (8)$$

The ratio $g(t)/g_0$ is thus periodic with period T_c ; and, under the validity of the previous assumption, its value is independent of experimental parameters such as the Rabi frequency and the modulation depth. It depends only on the ratio between the cycle time T_c and the interrogation time T . In a modern atomic frequency standard, such as an ion trap or an atomic fountain, the cycle duration is the sum of an unavoidable dead time plus the interrogation time; therefore the function $g(t)$ is not a constant during each cycle. As explained in the following and in companion papers [2], [3], this causes degradation of the frequency stability of the locked oscillator. Clearly, this very simple model neglects the response of the atoms to the field during the microwave interactions of duration τ_p .

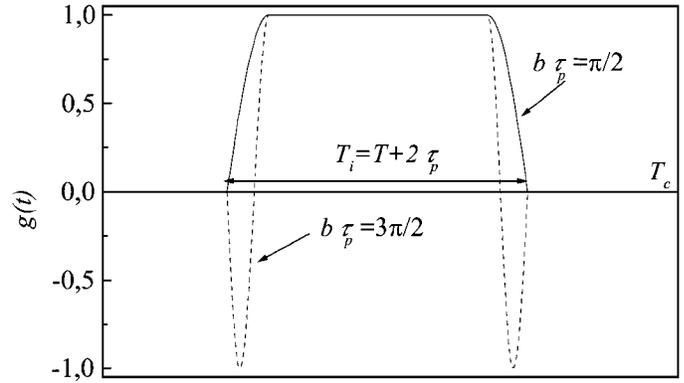


Fig. 1. The function $g(t)$, assumed centered in the cycle period, for the case of a Ramsey interrogation scheme with $b\tau_p = \pi/2$ (solid line) and $b\tau_p = 3\pi/2$ (dashed line), with $T = 0.5$ s, $T_c = 1$ s, $\tau_p = 15$ ms, and $\Omega_0 = -\omega_m$.

B. The Sensitivity Function for the Ramsey Interrogation

A more general approach to the calculation of $g(t)$ can be performed using the density matrix formalism for a two-level atom (see the Appendix). The function $g(t)$ can be calculated analytically for the Rabi or Ramsey interrogation scheme, if the Rabi frequency b is constant during the microwave pulses. We limit ourselves to the Ramsey case, which is commonly used. Under the conditions $T \gg \tau_p$ and $\Omega_0 \ll b$ we have:

$$g(t) = \begin{cases} a \sin bt & 0 \leq t \leq \tau_p \\ a \sin b\tau_p & \tau_p \leq t \leq T + \tau_p \\ a \sin b(T + 2\tau_p - t) & T + \tau_p \leq t \leq T + 2\tau_p \\ 0 & T + 2\tau_p \leq t \leq T_c \end{cases} \quad (9)$$

where $a = -\sin \Omega_0 T \sin b\tau$ and $\Omega_0 = \pm\omega_m$ according to the half period of modulation considered. Fig. 1 shows the variation of $g(t)$ for $b\tau_p = \pi/2$ and $b\tau_p = 3\pi/2$. Unlike the simple model of (6) the shape of $g(t)$ is strongly dependent on the microwave power applied during the microwave pulses.

Another case that is relevant is the atom-field interaction in a multi- λ cylindrical cavity, resonating in the TE_{01n} mode [5], which is used in the PHARAO prototype [6]. In this device, balls of cold cesium atoms will be launched along the axis of a cavity excited in such a mode. During their interrogation, the atoms experience a microwave field whose amplitude is proportional to $\sin(n\pi t/T_i)$, where T_i is the total interaction time. In this case, $g(t)$ must be calculated numerically (see the Appendix). Fig. 2 shows the variation of $g(t)$ for $n = 3$. Here, the operating parameters are chosen to provide the maximum slope of the resonance curve. This is achieved for $b_c T_i/n = 3.66$ and $\omega_m T_i = 2.31$, where b_c is the Rabi frequency at an anti-node of the microwave field.

It is clear that the shape of $g(t)$ depends on the type of interrogation scheme and on the details of the interaction such as field power and frequency detuning.

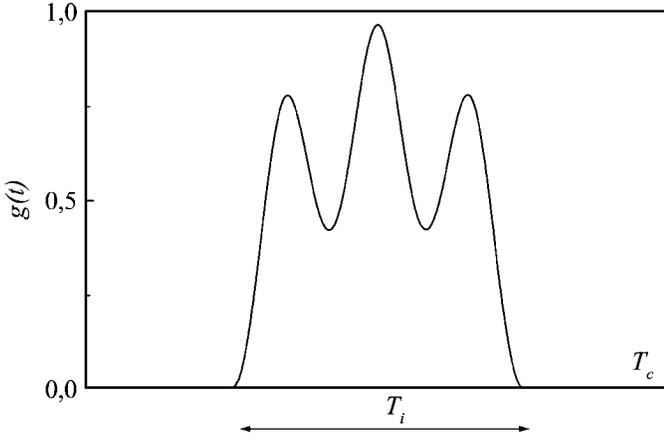


Fig. 2. The function $g(t)$, assumed centered in the cycle period, for the case of a multi- λ interrogation scheme in a TE_{013} cavity, with $T_i = 0.53$ s, $T_c = 1$ s, $b_c T_i/3 = 3.66$, $\omega_m T_i = 2.31$, and $\Omega_0 = -\omega_m$.

III. LIMITATION OF THE FREQUENCY STABILITY DUE TO SAMPLING

The control loop being closed, frequency corrections are applied to the interrogation oscillator at discrete times t_k , at the end of each cycle. It is possible to show that the spectral components of the interrogation oscillator phase noise around frequencies m/T_c are translated to frequencies below $1/T_c$ [2], [3]. This spectrum folding is at the origin of the frequency stability degradation of the atomic frequency standard. For very low Fourier frequencies, the down-converted noise spectrum can be assumed white.

The Allan variance of the locked interrogation oscillator is related to the frequency noise spectral density of the free running oscillator and to the harmonic content of the function $g(t)$ [2], [3] in the following way:

$$\sigma_{y \text{ lim}}^2(\tau) = \frac{1}{\tau} \sum_{m=1}^{\infty} \left(\frac{g_m^c}{g_0^c} + \frac{g_m^s}{g_0^s} \right) S_y^f(m/T_c) \quad (10)$$

where $\sigma_{y \text{ lim}}^2(\tau)$ is a lower limit to the achievable stability. Here $S_y^f(m/T_c)$ is the one-sided power spectral density of the relative frequency fluctuations of the free running interrogation oscillator at Fourier frequencies m/T_c , and the parameters g_0 , g_m^s , and g_m^c are defined by:

$$\begin{aligned} \begin{pmatrix} g_m^s \\ g_m^c \end{pmatrix} &= \frac{1}{T_c} \int_0^{T_c} g(\xi) \begin{pmatrix} \sin 2\pi m \xi / T_c \\ \cos 2\pi m \xi / T_c \end{pmatrix} d\xi, \\ g_0 &= \frac{1}{T_c} \int_0^{T_c} g(\xi) d\xi. \end{aligned} \quad (11)$$

It is possible without loss of generality to simplify (10) by applying a time translation to the function $g(t)$ in order to obtain a cosine series. Fig. 3 shows the coefficients $(g_m/g_0)^2 = ((g_m^s)^2 + (g_m^c)^2)/g_0^2$ versus the rank m for the function $g(t)$ in the Ramsey interrogation scheme for the two cases $b\tau_p = \pi/2$, which provides the optimal interrogation

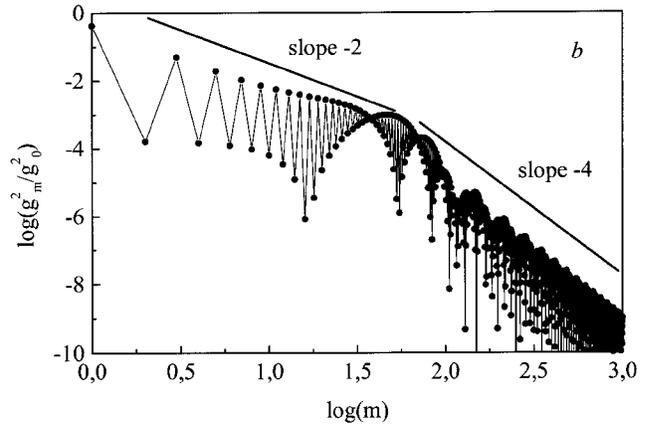
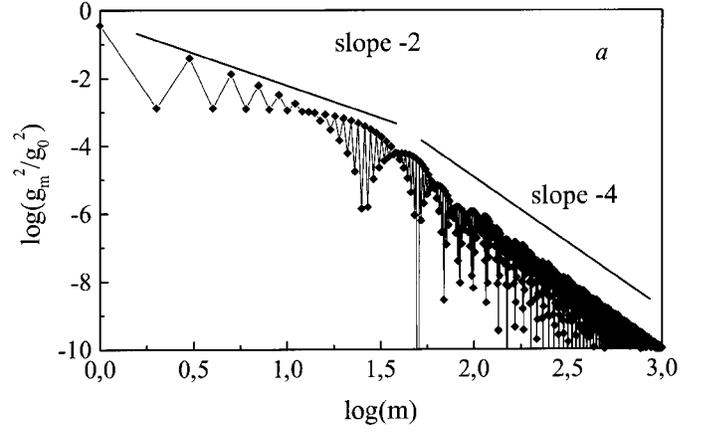


Fig. 3. Calculated spectrum of the function $g(t)/g_0$ for the case of Ramsey interrogation: (a) $b\tau_p = \pi/2$ (b) $b\tau_p = 3\pi/2$, with $T = 0.5$ s, $T_c = 1$ s, and $\tau_p = 15$ ms.

condition, and $b\tau_p = 3\pi/2$, which is used in the fountain frequency standard to evaluate the power dependent shifts. It is assumed that $\tau_p = 0.015$ s, $T = 0.5$ s, and $T_c = 1$ s. Fig. 4 shows the coefficients $(g_m/g_0)^2$ versus the rank m for the function $g(t)$ assuming interrogation in a multi- λ TE_{013} cavity, with $b_c T_i/3 = 3.66$ and $\omega_m T_i = 2.31$. It is worth noting that, for m larger than about 10, $(g_m/g_0)^2$ decreases as m^{-6} in this case. This property provides very good immunity against the white phase noise of the oscillator.

IV. EXPERIMENTAL EVALUATION OF THE FREQUENCY STABILITY DEGRADATION

In order to verify the model and provide evidence of the down-conversion effect, we have made various measurements with the interrogation oscillator used with the LPTFs Cs atomic fountain. We have purposely degraded its spectrum with different types of frequency noise. Fig. 5 is a schematic of the experimental set-up. We have used three different sources of noise: a white noise signal in the range of 0.1 Hz to 1600 Hz (f^0) with which various low-

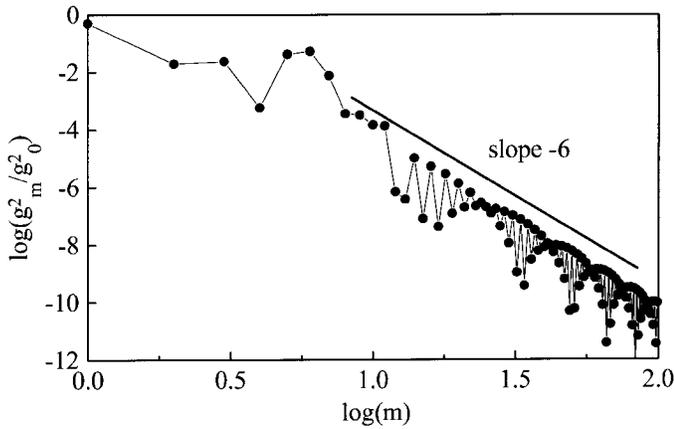


Fig. 4. Calculated spectrum of the function $g(t)/g_0$ for the case of multi- λ interrogation in a TE_{013} cavity, with $T_i = 0.53$ s, $T_c = 1$ s, $b_c T_i/3 = 3.66$, and $\omega_m T_i = 2.31$.

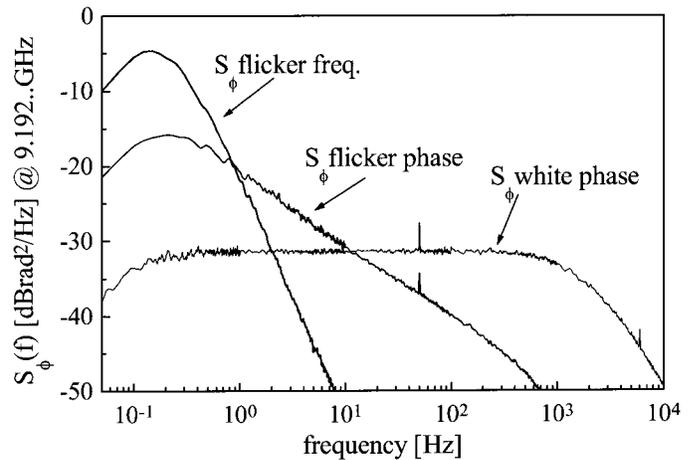


Fig. 6. Measured phase noise spectral density of the degraded interrogation oscillator for the three types of noise used in the experiment.

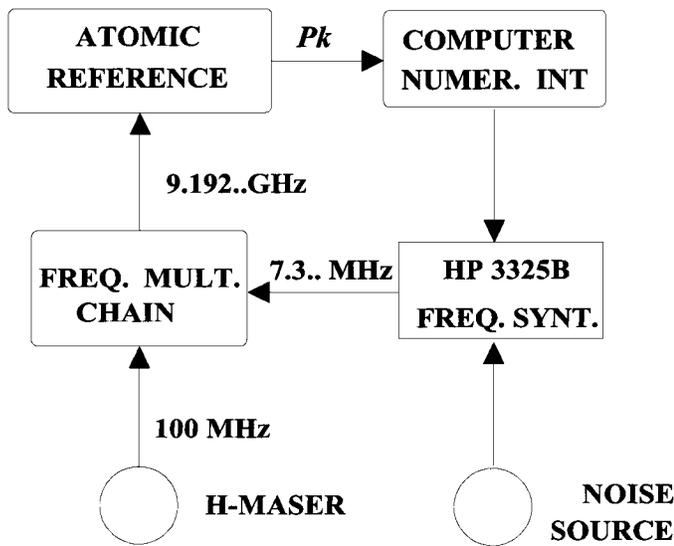


Fig. 5. Schematic of the atomic fountain frequency lock loop.

pass filters can be used; a flicker noise generator (f^{-1}) in the range of 0.5 to 100 Hz; and a generator with spectral density proportional to f^{-3} , for Fourier frequencies from 0.5 to 100 Hz. Fig. 6 shows the corresponding phase noise power spectral densities. We use the noise generators to drive the phase modulation input of an offset synthesizer. The phase noise added to the synthesizer is transferred to the interrogation oscillator spectrum at 9.192 GHz.

It is worth noting that, in our measurement set-up, the interrogating oscillator is obtained by phase-locking a frequency multiplication chain to the reference H-maser, as shown in Fig. 5. The frequency-locking of the interrogation oscillator to the atomic resonance is obtained by controlling the central frequency of the offset synthesizer used to generate the difference between the 92nd harmonic of the 100 MHz H-maser signal and the hyperfine frequency of the cesium atom.

The frequency stability of the atomic fountain measured against the hydrogen maser is then obtained by calculating the Allan standard deviation on the frequency corrections

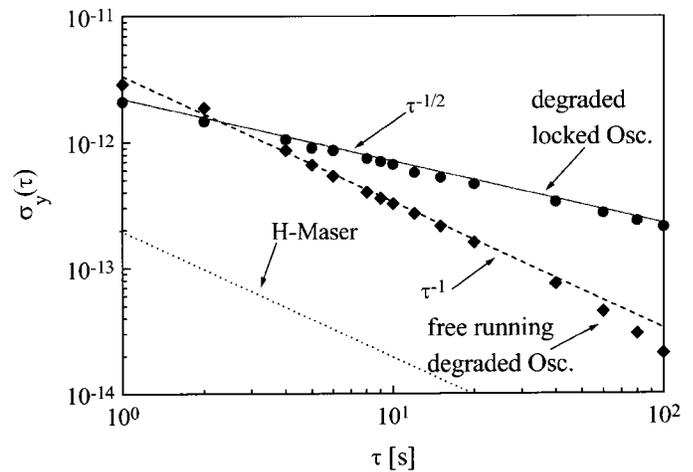


Fig. 7. Fractional frequency stability of the free running and of the locked interrogation oscillator.

applied to the offset synthesizer. As a consequence, the interrogating oscillator has to be degraded only for frequencies larger than or equal to the cycle frequency. For low frequencies (i.e., for times long compared to the cycle period), the noise sources are high pass filtered. This avoids the long-term degradation and is particularly effective for the flicker frequency noise which gives a flat Allan variance. As shown in Fig. 7, the fractional frequency stability of the free running oscillator behaves as τ^{-1} , whereas the stability of the locked oscillator is proportional to $\tau^{-1/2}$. This clearly shows that the frequency stability of the locked oscillator is dominated by the aliasing noise for integration times longer than 10 to 20s. We measured the stability for two conditions: $b\tau_p = \pi/2$ and $b\tau_p = 3\pi/2$. Tables I and II report the calculated values using (10) and the measured data for the flicker phase and flicker frequency noises. For these colored noises, (10) shows that the frequency stability is mainly limited by the first term of the series, which depends only on the ratio between the interrogation time T_i and the cycle time T_c . In order to verify precisely the model, we need a measurement that is more sensitive to

TABLE I
 $b\tau_p = \pi/2$

Type of noise	σ_y meas.(1s)	σ_y calc.(1s)
Frequency flicker	$2.4 \cdot 10^{-12}$	$2.3 \cdot 10^{-12}$
Phase flicker	$3.0 \cdot 10^{-12}$	$2.9 \cdot 10^{-12}$

TABLE II
 $b\tau_p = 3\pi/2$

Type of noise	σ_y meas.(1s)	σ_y calc.(1s)
Frequency flicker	$2.8 \cdot 10^{-12}$	$2.4 \cdot 10^{-12}$
Phase flicker	$4.8 \cdot 10^{-12}$	$4.6 \cdot 10^{-12}$

the shape of $g(t)$. In the case of white phase noise, the frequency noise spectrum behaves as f^2 and consequently the down-conversion is strongly dependent on the harmonic content of $g(t)/g_0$. Changing the white phase noise spectrum by low-pass filtering the noise source with different cut-off frequencies improves the sensitivity of the measurements. Fig. 8 reports the calculated and measured results, which agree within the limits of the measurement errors, estimated to be about 20%. Measurements performed with the filtered white phase noise confirm the validity of the model, even for different interrogation oscillator levels, i.e., for $b\tau_p = \pi/2$ and $3\pi/2$. It is interesting to note that, for large filter cut-off frequency, the degradation ratio is equal to 3. For the colored noises the degradation does not depend on the microwave power.

V. DISCUSSION

As shown before, the degradation depends on the harmonic content of $g(t)$. A constant value of the sensitivity function over the cycle would eliminate this effect. Unfortunately, the operation of the fountain frequency standard requires unavoidable dead times.

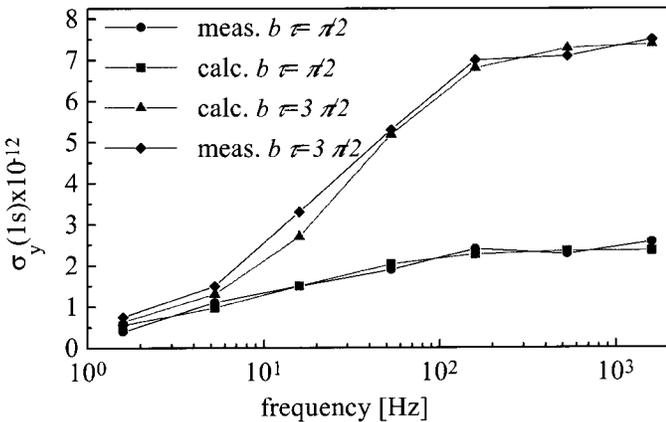


Fig. 8. Measured and calculated frequency stability versus the cut-off frequency of the white phase noise for $b\tau_p = \pi/2$ (circles, squares) and $b\tau_p = 3\pi/2$ (diamonds, triangles).

The level of the effect would obviously be reduced with an oscillator, showing a much improved spectral purity, such as a cryogenic sapphire oscillator [7], [8]. A reduction of this “intermodulation” noise has already been obtained in a Rb cell standard [9] and in a thermal Cs beams clock [10] by filtering at $1/T_c$ the frequency noise of the interrogation oscillator. In a cesium fountain, in the PHARAO set-up or in a trapped ion frequency standard, it will not be feasible to use notch filters to reject the oscillator noise at such low Fourier frequencies (≤ 1 Hz).

If we consider the state of the art of quartz oscillators, the phase noise below a few Hertz is mainly limited by flicker frequency noise. In this case it is easy to obtain an approximate relationship between the flicker floor of the interrogation oscillator σ_y^{LO} and the frequency stability of the standard $\sigma_{y\text{lim}}(\tau)$ versus the duty cycle, defined as $d = T/T_c$. In the case of the Ramsey interrogation with $T \gg \tau_p$ and $0.4 < d < 0.7$ we have:

$$\begin{aligned} \sigma_{y\text{lim}}(\tau) &\cong \frac{\sigma_y^{LO}}{\sqrt{2\ln(2)}} \left| \frac{g_1}{g_0} \right| \sqrt{\frac{T_c}{\tau}} \\ &= \frac{\sigma_y^{LO}}{\sqrt{2\ln(2)}} \left| \frac{\sin(\pi d)}{\pi d} \right| \sqrt{\frac{T_c}{\tau}}. \end{aligned} \quad (12)$$

Apparently the only way to reduce this detrimental effect is to increase the duty cycle. For a given duty cycle, the degradation is proportional to $\sqrt{T_c}$. This result is most significant for trapped ions standards and for the PHARAO clock, where $T_c \sim 3\text{--}10$ s.

To illustrate the beneficial effect of the increase of the interrogation duty cycle d , or of the release of several balls during each cycle in the case of the PHARAO clock in space, we have made some numerical calculations, in which the quartz oscillator is assumed to show a frequency noise spectral density given by:

$$S_y^f(f) = 3.2 \cdot 10^{-29} f^2 + 1.0 \cdot 10^{-27} f + 3.2 \cdot 10^{-26} / f. \quad (13)$$

In the case of the Ramsey method of interrogation, the total interrogation time is $T_i = T + 2\tau_p$. Fig. 9 shows the variation of $\sigma_{y\text{lim}}$ versus T_i/T_c for three sets of parameter values, and Fig. 10 shows the variation of $\sigma_{y\text{lim}}$ versus the number of balls. It is assumed that the interrogation occurs in a TE_{013} cavity and that the time interval between two successive ball releases is $T_i/6$. Again, three sets of parameter values are considered.

Thus it is possible, in an atomic resonator based on the interrogation of atoms launched sequentially, to reduce the limiting effect we have considered. This may be accomplished by a proper design of the resonator leading to as large as possible duty cycle and/or by launching several clouds of atoms during one cycle. For the case of ion traps, the use two parallel traps has been proposed [11].

One may also note that, in the two examples given, $\sigma_{y\text{lim}}$ is smallest for $1/T_c = 1$ Hz. This value is the close to the Fourier frequency, f_Q , for which $S_y^f(f)$ of the VCXO shows a minimum. With the data of (11), we have

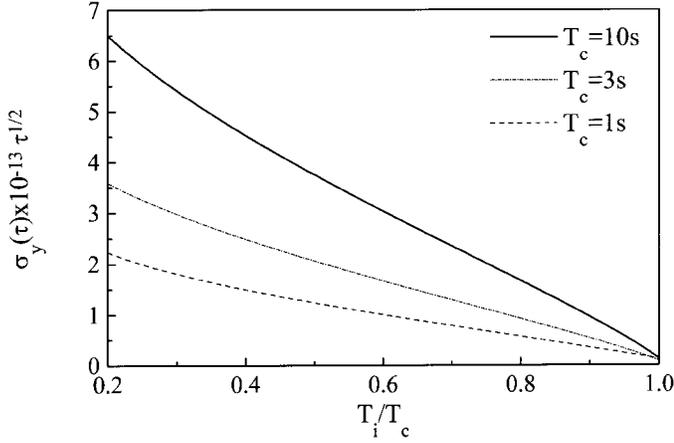


Fig. 9. Frequency stability versus the interrogation duty cycle for the Ramsey interrogation for different cycle lengths. With $T_i = 0.5T_c$ and $\tau_p = 0.015T_c$.

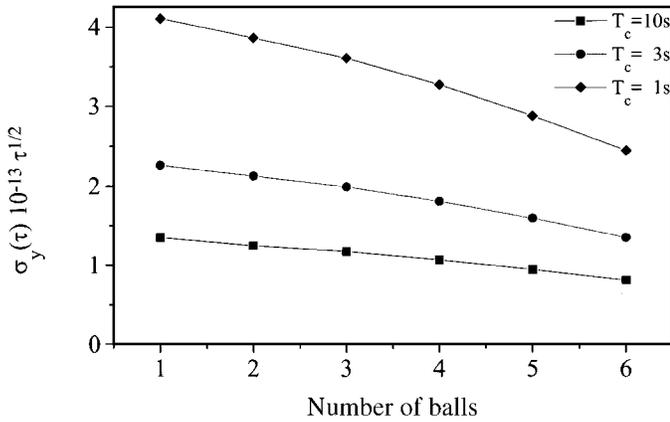


Fig. 10. Calculated frequency stability versus the number of launched atomic balls for the multi- λ case. The atomic balls are separated by $T_i/6$ and $T_i = 0.53T_c$, $b_c T_i/3 = 3.66$, and $\omega_m T_i = 2.31$.

$f_Q = 5$ Hz. This suggests that, whenever possible, the characteristics of the atomic resonator and of the VCXO should be matched. This is achieved when the condition $f_Q \approx 1/T_c$ is fulfilled.

VI. CONCLUSIONS

In this paper we have developed a quantum mechanical calculation of the atomic response to the phase noise of the interrogation oscillator in a two level atom. This model has been used in the calculation of the frequency stability degradation in a pulsed atomic frequency standard due to the down-conversion of the frequency noise of the interrogating oscillator. We also have compared results of calculations based on this model with experimental values obtained by using the LPTF Cs atomic fountain with a purposely degraded oscillator. The theory and the experiments agree within the limits of measurement errors. For a state-of-the-art 5 or 10 MHz BVA quartz oscillator, the excess noise due to the sampling process limits the frequency stability of an atomic fountain to about $10^{-13} \tau^{-1/2}$

for a cycle time of 1 s. Better results could be achieved using cryogenic sapphire oscillators. The model shows that the limitation comes primarily from the flicker frequency noise of the oscillator and that the characteristic white phase floor does not affect the results.

There do not seem to be any obvious signal processing techniques that could be applied to reduce the consequences of this detrimental effect.

APPENDIX

As shown in [12], the change of the quantum state of the atoms interacting with a quasi-resonant field $b \cos(\omega t + \Delta\phi)$ can be represented in a matrix form. We have, in general,

$$\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix} = \mathcal{R}[b, \Delta\phi, \Omega_0, t] \begin{pmatrix} a_1(0) \\ a_2(0) \\ a_3(0) \end{pmatrix}, \quad (\text{A-1})$$

where $a_1(t)$ and $a_2(t)$ denote the atomic coherence and $a_3(t)$ the relative population difference of the two levels involved in the transition. The column matrices at the right and at the left represent the atom properties at the beginning and at the end of an interaction of duration t , respectively. $\mathcal{R}[b, \Delta\phi, \Omega_0, t]$ is a 3×3 matrix whose elements depend on t , the Rabi frequency b , the phase ϕ , and the detuning from the atomic resonance. We have [see (A-2) top of next page]:

Therefore, the population difference of atoms submitted to various amplitude and phase conditions during their interaction with the magnetic microwave field can be calculated from matrix products. The variation of the probability that a transition took place is related to the matrix elements in the following way:

$$P(t, \Delta\phi) = \frac{1}{2} \left(1 - \frac{a_3(t, \Delta\phi)}{a_3(0)} \right). \quad (\text{A-3})$$

According to (1) the sensitivity function $g(t)$ is:

$$\begin{aligned} g(t) &= 2 \lim_{\Delta\phi \rightarrow 0} \delta P(t, \Delta\phi) / \Delta\phi \\ &= \left. \frac{\partial a_3(t, \Delta\phi)}{\partial \Delta\phi} \right|_{\Delta\phi=0} \frac{1}{a_3(0)}. \end{aligned} \quad (\text{A-4})$$

The effect of the small phase step $\Delta\phi$ occurring at a given time during the interaction can be expressed easily. In the case of a Ramsey interrogation where the Rabi frequency b is constant, we can write:

$$\begin{aligned} a(t, T, \tau_p, \Delta\phi, b, \Omega_0)_1 &= \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p] \mathcal{R}[0, \Delta\phi, \Omega_0, T] \\ &\quad \times \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p - t] \mathcal{R}[b, 0, \Omega_0, t] a(0) \end{aligned}$$

if the phase step takes place during the first microwave interaction;

$$\begin{aligned} a(t, T, \tau_p, \Delta\phi, b, \Omega_0)_2 &= \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p] \mathcal{R}[0, 0, \Omega_0, T] \\ &\quad \times \mathcal{R}[b, 0, \Omega_0, \tau_p] a(0) \end{aligned}$$

$$\mathcal{R}[b, \Delta\phi, \Omega_0, t] = \begin{pmatrix} \begin{pmatrix} \cos \Omega t + \\ \frac{b^2 \cos^2 \Delta\phi}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} & \begin{pmatrix} -\frac{\Omega_0}{\Omega} \sin \Omega t + \\ -\frac{b^2 \cos \Delta\phi \sin \Delta\phi}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} & \begin{pmatrix} -\frac{b \cos \Delta\phi \Omega_0}{\Omega^2} (1 - \cos \Omega t) \\ + \frac{b \sin \Delta\phi}{\Omega} \sin \Omega t \end{pmatrix} \\ \begin{pmatrix} \frac{\Omega_0}{\Omega} \sin \Omega t + \\ -\frac{b^2 \cos \Delta\phi \sin \Delta\phi}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} & \begin{pmatrix} \cos \Omega t + \\ \frac{b^2 \sin^2 \Delta\phi}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} & \begin{pmatrix} \frac{b \cos \Delta\phi}{\Omega} \sin \Omega t + \\ \frac{b \Omega_0 \sin \Delta\phi}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} \\ \begin{pmatrix} -\frac{b \cos \Delta\phi \Omega_0}{\Omega^2} (1 - \cos \Omega t) \\ -\frac{b \sin \Delta\phi}{\Omega} \sin \Omega t \end{pmatrix} & \begin{pmatrix} -\frac{b \cos \Delta\phi}{\Omega} \sin \Omega t \\ + \frac{b \sin \Delta\phi \Omega_0}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} & \begin{pmatrix} 1 - \frac{b^2}{\Omega^2} (1 - \cos \Omega t) \end{pmatrix} \end{pmatrix},$$

$$\Omega = \sqrt{b^2 + \Omega_0^2}. \quad (\text{A-2})$$

if the phase step takes place during the period T ; and

$$a(t, T, \tau_p, \Delta\phi, b, \Omega_0)_3 = \mathcal{R}[b, \Delta\phi, \Omega_0, \tau_p - t] \mathcal{R}[b, 0, \Omega_0, t] \\ \times \mathcal{R}[0, 0, \Omega_0, T] \mathcal{R}[b, 0, \Omega_0, \tau_p] a(0)$$

if the phase step takes place during the last microwave interaction. We suppose that the atomic population is prepared in a single quantum state at the beginning of the interaction, i.e., $a_1(0) = a_2(0) = 0$, $a_3(0) = -1$. At the end of the interaction we measure the relative population difference $a_3(t)$. The function $g(t)$ can be easily obtained by expanding $a_3(t)$ to first order with respect to $\Delta\phi$. We thus have:

$$g(t) = \begin{cases} \left. \frac{\partial}{\partial \phi} a_3(t, T, \tau_p, \Delta\phi, b, \Omega_0)_1 \right|_{\Delta\phi=0} & 0 \leq t \leq \tau_p \\ \left. \frac{\partial}{\partial \phi} a_3(t, T, \tau_p, \Delta\phi, b, \Omega_0)_2 \right|_{\Delta\phi=0} & \tau_p \leq t \leq T + \tau_p \\ \left. \frac{\partial}{\partial \phi} a_3(t, T, \tau_p, \Delta\phi, b, \Omega_0)_3 \right|_{\Delta\phi=0} & T + \tau_p \leq t \leq T + 2\tau_p \\ 0 & T + 2\tau_p \leq t \leq T_c. \end{cases} \quad (\text{A-5})$$

It is worth noting that, after the end of the interaction, and up to the end of the interrogation cycle T_c the value of the function is null. With the conditions $T \gg \tau_p$ and $\Omega_0 \ll b$ we obtain:

$$g(t) = \begin{cases} a \sin bt & 0 \leq t \leq \tau_p \\ a \sin b\tau_p & \tau_p \leq t \leq T + \tau_p \\ a \sin b(T + 2\tau_p - t) & T + \tau_p \leq t \leq T + 2\tau_p \\ 0 & T + 2\tau_p \leq t \leq T_c, \end{cases} \quad (\text{A-6})$$

where the constant a has the same meaning as in (9). When the microwave amplitude is not a constant during the interaction, two different methods can be implemented. One may divide the interaction time into elementary intervals during which the amplitude is assumed a constant, or the differential equations describing the evolution of $a_1(t')$, $a_2(t')$, and $a_3(t')$ can be integrated numerically. In the case of atoms traveling along the axis of a TE_{01n} microwave cavity, and, assuming that the phase step $\Delta\phi$ is very small

[12], these equations take the form:

$$\frac{\partial a_1(\xi)}{\partial \xi} + \frac{\Omega_0 T_i}{n} a_2(\xi) - b_c \Delta\phi \frac{T_i}{n} a_3(\xi) \sin(\pi\xi) = 0 \\ \frac{\partial a_2(\xi)}{\partial \xi} - \frac{\Omega_0 T_i}{n} a_1(\xi) - b_c \Delta\phi \frac{T_i}{n} a_3(\xi) \sin(\pi\xi) = 0 \\ \frac{\partial a_3(\xi)}{\partial \xi} + \frac{b_c T_i}{n} (a_1(\xi) \Delta\phi + a_2(\xi)) \sin(\pi\xi) = 0, \quad (\text{A-7})$$

with:

$$\xi = \frac{nt'}{T_i}. \quad (\text{A-8})$$

In these equations, T_i is the time of flight across the cavity and b_c is the Rabi frequency at an antinode point of the microwave field. The value of the frequency sensitivity function $g(t)$, at the time t is computed by integrating the differential system with $\Delta\phi = 0$ for $0 < \xi < nt/T_i$ and $\Delta\phi \neq 0$ for $nt/T_i < \xi < n$. The change of the population difference is then calculated at the outcome of the atom-field interaction and $g(t)$ is obtained from (A-4).

In the case of multi-ball operation, the total sensitivity function $g_{mb}(t)$ is obtained in the following way:

$$g_{mb}(t) = \sum_{j=1}^{n_b} g(t - j\Delta t), \quad (\text{A-9})$$

where n_b is the number of released atomic balls and Δt is the time interval between two successive balls.

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A Derivation of the Long-Term Degradation of a Pulsed Atomic Frequency Standard from a Control-Loop Model

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Abstract—The phase of a frequency standard that uses periodic interrogation and control of a local oscillator (LO) is degraded by a long-term random-walk component induced by downconversion of LO noise into the loop passband. The Dick formula for the noise level of this degradation is derived from an explicit solution of an LO control-loop model.

I. INTRODUCTION

IN 1987, following a suggestion of L. Cutler, Dick [1] described a source of long-term instability for a class of passive frequency standards that includes ion traps and atomic fountains. In these standards, the frequency of a local oscillator (LO) is controlled by a feedback loop whose detection and control operations are periodic with some period T_c . For each cycle, the output of the detector is a weighted average of the LO frequency error over the cycle. The weighting function $g(t)$, derived from quantum-mechanical calculations not addressed here, depends on the method by which the atoms are interrogated by the radio-frequency field generated by upconversion of the LO signal to the atomic transition frequency [1]–[4]. In general, $g(t)$ can be zero over a considerable portion of the cycle. The level of the LO control signal over a cycle is a function of the detector outputs from previous cycles.

A frequency-control loop works by attenuating the frequency fluctuations of the LO inside the loop passband (long-term fluctuations), while tolerating them outside the passband (short-term fluctuations). As Dick saw, though, the periodic interrogation causes out-of-band LO noise power, near the cycle frequency $f_c = 1/T_c$ and its harmonics, to be downconverted into the loop passband, thus injecting random false information about the current average LO frequency into the control signal. This random false frequency correction causes a component of white frequency modulation (FM), or random walk of phase, to persist in the output of the locked LO (LLO) over the long term. Dick gave the formula

$$S_y(0) = 2 \sum_{k=1}^{\infty} \frac{g_k^2}{g_0^2} S_y^{\text{LO}}(kf_c) \quad (1)$$

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for the white-FM noise level contributed by this effect. Here, $S_y^{\text{LO}}(f)$ is the spectral density of the normalized frequency departure of the free-running LO, and g_k is the Fourier coefficient

$$g_k = \frac{1}{T_c} \int_0^{T_c} g(t) \cos(2\pi k f_c t) dt, \quad (2)$$

where $g(t)$ is assumed to be symmetric about $T_c/2$. This level of white FM near Fourier frequency zero contributes an asymptotic component of Allan variance given by

$$\sigma_y^2(\tau) \sim \frac{S_y(0)}{2\tau} \quad (f_c \tau \rightarrow \infty).$$

The purpose of the present paper is to supplement previous derivations [1]–[3], [5] of the Dick formula (1) by an approach that uses an explicit time-domain solution of a simple LO control loop model with a general detection weighting function $g(t)$. Careful interpretation of this solution yields a formula for the LLO frequency spectrum, and conditions for the validity of the Dick formula. The model treated below is not intended as a realistic representation of an actual frequency standard; the goal is to improve understanding of the Dick effect by exhibiting its presence in a model with minimal features. Similar models have been treated by Audoin *et al.* [6], who use an equivalent time-domain solution method, and by Lo Presti *et al.* [5], [7], who use a Fourier transform method. The Lo Presti model also has been treated by the time-domain method [8].

II. CONTROL-LOOP MODEL

Fig. 1 shows the chosen model for an LO control loop, containing both analog and digital elements. All signals are scaled as normalized frequency departure from the ideal frequency determined by the atomic transition. The frequency noise contributed by the free-running local oscillator is $y_{\text{LO}}(t)$. The output LLO frequency is $y(t)$. The error signal

$$\frac{1}{T_c g_0} \int_0^{T_c} g(u) y((n-1)T_c + u) du \quad (3)$$

from the interrogation of $y(t)$ during the n th cycle $(n-1)T_c < t < nT_c$ is implemented in Fig. 1 by a linear time-invariant filter G with the normalized time-reversed

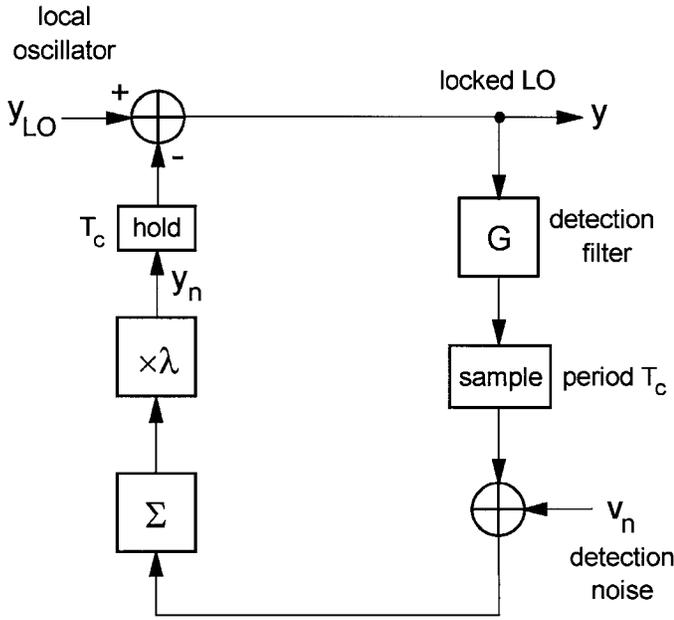


Fig. 1. A feedback-loop model for a local oscillator with periodic interrogation and control. The impulse response of the filter G is a normalized, time-reversed version of the interrogation sensitivity function $g(t)$.

impulse-response function

$$g_1^-(t) = \frac{g(T_c - t)}{T_c g_0}, \quad 0 < t < T_c$$

$$= 0 \quad \text{otherwise}$$

and transfer function

$$G(f) = \int_0^{T_c} g_1^-(t) e^{-i2\pi ft} dt.$$

The symmetry assumption about $g(t)$ has been dropped. The filter output

$$Gy(t) = \int_0^{T_c} g_1^-(u) y(t - u) du,$$

sampled at time $t = nT_c$, is exactly (3). The detection noise term v_n can represent photon-count fluctuations in frequency standards with optical detection, for example. The cumulative sum of the error signals, multiplied by a gain factor λ , is the frequency correction y_n , which is applied to the LO during the next cycle $nT_c < t < (n+1)T_c$. Except for initial conditions, the following two equations define the closed-loop model completely:

$$y_n = y_{n-1} + \lambda(Gy(nT_c) + v_n), \quad (4)$$

$$y(t) = y_{LO}(t) - y_{n-1}, \quad (n-1)T_c < t < nT_c, \quad (5)$$

in which it is convenient to suppose that n runs through all integers. This system has two inputs, $y_{LO}(t)$ and v_n , and one output, $y(t)$.

III. THE LLO FREQUENCY

The mixed analog-digital system (4), (5) can be solved by eliminating $y(t)$ to get an equation in y_n alone. From (5) we have $Gy(nT_c) = Gy_{LO}(nT_c) - y_{n-1}$, which, substituted into (4), gives the first-order difference equation:

$$y_n = (1 - \lambda)y_{n-1} + \lambda w_n, \quad (6)$$

where

$$w_n = Gy_{LO}(nT_c) + v_n. \quad (7)$$

Assume $0 < \lambda < 1$. Then the general solution of (6) is

$$y_n = \sum_{j=0}^{\infty} \lambda(1-\lambda)^j w_{n-j} + C(1-\lambda)^n. \quad (8)$$

From now on we shall ignore the transient part of this solution by setting $C = 0$.

Let us express y_n directly as a function of the inputs $y_{LO}(t)$ and v_n . Define the discrete-time lowpass filter H_d with weights

$$h_j = \lambda(1-\lambda)^j, \quad j \geq 0,$$

which sum to 1, and transfer function

$$H_d(z) = \sum_{j=0}^{\infty} h_j z^{-j} = \frac{\lambda}{1 - (1-\lambda)z^{-1}}, \quad (9)$$

where, from now on, $z = e^{i2\pi f T_c}$. Substituting (7) into (8) gives

$$y_n = \int_0^{\infty} h_c(t) y_{LO}(nT_c - t) dt + H_d v_n,$$

$$= H_c y_{LO}(nT_c) + H_d v_n, \quad (10)$$

in which we have introduced a causal continuous-time filter H_c with impulse response

$$h_c(t) = \sum_{j=0}^{\infty} h_j g_1^-(t - jT_c)$$

consisting of repetitions of $g_1^-(t)$ with exponentially decreasing amplitudes. Notice that $\int_0^{\infty} h_c(t) dt = 1$. Its transfer function

$$H_c(f) = \int_0^{\infty} h_c(t) e^{-i2\pi ft} dt$$

satisfies

$$H_c(f) = H_d(z)G(f). \quad (11)$$

Substituting (10) into (5) gives an explicit piecewise solution for the LLO frequency:

$$y(t) = y_{LO}(t) - H_c y_{LO}((n-1)T_c) - H_d v_{n-1},$$

$$(n-1)T_c < t < nT_c. \quad (12)$$

IV. THE LLO FREQUENCY SPECTRUM

Although (12) gives an explicit formula for the output frequency, its interpretation requires careful handling. Under reasonable assumptions (see below) on $y_{\text{LO}}(t)$ and v_n as random processes, we cannot expect the piecewise-defined process $y(t)$ to be stationary, or even to have stationary n th increments for some n . Thus, the author does not know how to assign a spectral density to it. To get around this problem, it is convenient to study the samples $x(nT_c)$ of the LLO time residual $x(t) = \int y(t) dt$. The properties of these samples are determined in turn by the properties of the average LLO frequencies

$$\begin{aligned} Ay(nT_c) &= \frac{1}{T_c} \int_{(n-1)T_c}^{nT_c} y(t) dt \\ &= \frac{x(nT_c) - x((n-1)T_c)}{T_c} \end{aligned}$$

where A is the moving-average filter whose action on a signal $\xi(t)$ is

$$A\xi(t) = \frac{1}{T_c} \int_0^{T_c} \xi(t-u) du.$$

Its transfer function is

$$A(f) = \frac{1 - z^{-1}}{i2\pi f T_c}.$$

Applying A to (12) gives

$$\begin{aligned} Ay(nT_c) &= Ay_{\text{LO}}(nT_c) \\ &\quad - H_c y_{\text{LO}}((n-1)T_c) - H_d v_{n-1}. \end{aligned} \quad (13)$$

We are now going to derive the spectrum of the discrete-time process $Ay(nT_c)$ defined by (13). To this end, consider the auxiliary process defined by

$$Y(t) = Ay_{\text{LO}}(t) - H_c y_{\text{LO}}(t - T_c),$$

which is obtained from $y_{\text{LO}}(t)$ by a linear time-invariant filter B with transfer function

$$B(f) = A(f) - z^{-1} H_d(f) G(f). \quad (14)$$

Assume that $y_{\text{LO}}(t)$ is a mean-continuous random process with stationary first increments and a two-sided (even) spectral density $S_y^{\text{LO}}(f)$, which necessarily satisfies

$$\int_0^{f_c} S_y^{\text{LO}}(f) f^2 df < \infty, \quad \int_{f_c}^{\infty} S_y^{\text{LO}}(f) df < \infty \quad (15)$$

[9]. The first condition in (15) allows any power law spectrum $|f|^\alpha$ for $\alpha > -3$; linear combinations of such spectra constitute the spectra that are customarily attributed to oscillators. For $\alpha \geq -1$, the second condition in (15) requires a high-frequency rolloff of the $|f|^\alpha$ behavior.

The assumption (15) makes the process $Y(t)$ stationary: because $A(f)$, $H_d(z)$, and $G(f)$ are all $1 + O(f)$ as

$f \rightarrow 0$, we see from (14) that $B(f) = O(f)$. Thus $|B(f)|^2$ attenuates any low-frequency divergence of $S_y^{\text{LO}}(f)$ allowed by (15), leaving an integrable two-sided spectral density

$$S_Y(f) = |A(f) - z^{-1} H_d(z) G(f)|^2 S_y^{\text{LO}}(f). \quad (16)$$

The first two terms of the right side of (13) are just the samples $Y(nT_c)$, which constitute a discrete-time stationary process whose two-sided spectral density is

$$\sum_{k=-\infty}^{\infty} S_Y(f + kf_c), \quad |f| \leq f_c/2.$$

The terms with $k \neq 0$ account for the Dick effect. Let the detection noise process v_n be independent of $y_{\text{LO}}(t)$ and stationary, with two-sided spectral density $S_v(f)$. Then the process $Ay(nT_c)$ given by (13) is a stationary discrete-time process with two-sided spectral density

$$S_{Ay}(f) = S_{Ay}^0(f) + S_{Ay}^1(f), \quad |f| \leq f_c/2,$$

where

$$\begin{aligned} S_{Ay}^0(f) &= |A(f) - z^{-1} H_d(z) G(f)|^2 S_y^{\text{LO}}(f) \\ &\quad + |H_d(z)|^2 S_v(f), \end{aligned} \quad (17)$$

the main part, so to speak, and

$$\begin{aligned} S_{Ay}^1(f) &= \sum_{k \neq 0} \left| \frac{1 - z^{-1}}{i2\pi(fT_c + k)} - z^{-1} H_d(z) G(f + kf_c) \right|^2 \\ &\quad \times S_y^{\text{LO}}(f + kf_c), \end{aligned} \quad (18)$$

the aliased part, where the sum includes both positive and negative k .

An example of these frequency spectra is shown in Fig. 2, in which $T_c = 1$ s, $S_y^{\text{LO}}(f) = |f|^{-1}$ (flicker FM)¹, $g(t) = 1$ for $T_c/2 < t < T_c$ and 0 otherwise, and $\lambda = 1/10$. Detection noise is omitted. The spectra are plotted up to frequency $f_c/2$. Harmonics through order 5 are used to approximate the series in (18). Despite the attenuation of the main part of the LLO spectrum from the LO spectrum below the loop bandwidth, the white-FM contribution of the aliased part is dominant only for frequencies below $10^{-4} f_c$. The bandwidth of the aliased white-FM noise is approximately the same as the loop bandwidth.

V. THE DICK FORMULA

In general, the aliased part (18) of the LLO frequency spectrum introduces a long-term white-FM spectral component if (18) is continuous and positive at $f = 0$. Reasonable mathematical conditions on the weighting function and LO frequency spectrum will guarantee the continuity

¹Violation of the second condition in (15) does not really matter.

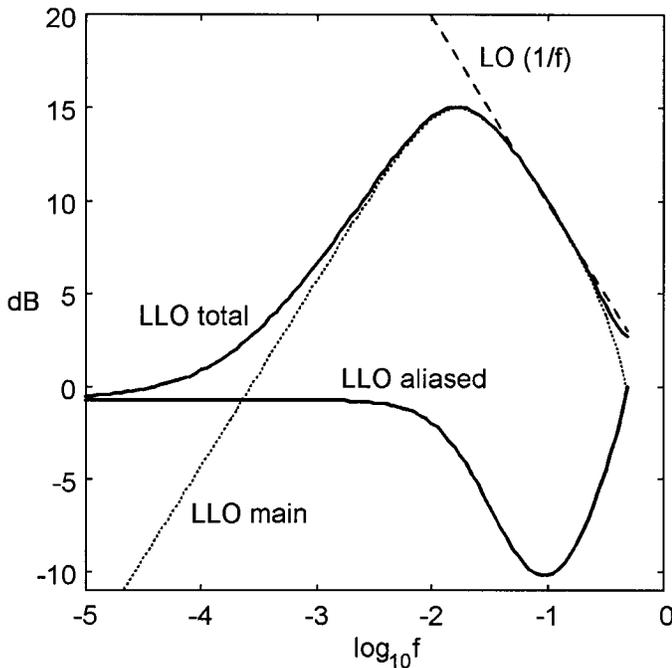


Fig. 2. Frequency spectra for a simple example with flicker-FM local-oscillator noise and a 1-second cycle. The aliased spectrum at low frequencies is responsible for the Dick effect.

of the aliased spectrum. For example, if $g(t)$ is square-integrable for $0 < t < T_c$, and $S_y^{\text{LO}}(f)$ is continuous and bounded for $|f| \geq f_c/2$, then it can be proved that the right side of (18) is a uniformly convergent series of continuous functions on $|f| \leq f_c/2$. Consequently, the sum of the series is a continuous function. Letting $f = 0$ in (18) gives

$$S_{A_y}^1(0) = 2 \sum_{k=1}^{\infty} \left| G(kf_c) \right|^2 S_y^{\text{LO}}(kf_c). \quad (19)$$

For the example of Fig. 2, $S_{A_y}^1(0) = (8/\pi^2) \sum_{j=0}^{\infty} (2j+1)^{-3} = 0.853$. The formula (19) holds for one-sided spectral densities also. Finally, if $g(t)$ is symmetric about $T_c/2$, then $G(kf_c) = g_k/g_0$. Thus (19) reduces to the Dick formula (1).

VI. REMARKS

The Dick effect can be hidden by the main part (17) of the LLO spectrum. If the detection noise v_n is white, then the term $|H_d(z)|^2 S_v(f)$ competes directly with the Dick effect as another white-FM noise at low frequencies. The basic action of the control loop operates on the LO frequency by a filter with frequency response

$|A(f) - z^{-1}H_c(z)|^2$, which, as we observed, is $O(f^2)$ as $f \rightarrow 0$. Thus, the filter adds 2 to the exponent of any low-frequency power law that $S_y^{\text{LO}}(f)$ obeys. If $S_y^{\text{LO}}(f)$ is more divergent than f^{-2} (random walk FM), then $S_{A_y}^0(f)$ is unbounded near $f = 0$, hence masks the Dick effect. Random walk FM in the LO is transformed to another white FM component in the LLO. Anything less divergent, like flicker FM, is transformed to an LLO spectral density that tends to zero at low frequencies. In this case, the Dick effect and the detection noise predominate in the long term.

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A Simple Analysis of the Dick Effect in Terms of Phase Noise Spectral Densities

Letizia Lo Presti, Daniele Rovera, *Member, IEEE*, and Andrea De Marchi

Abstract—In cold-atom frequency standards based on the Ramsey double interaction method, the phase noise of the interrogating signal appears as a random “end-to-end phase difference,” thereby introducing frequency noise in the loop. This phenomenon is analyzed in this paper in the Fourier frequency domain, using phase noise power spectral densities $S_\phi(f)$. In continuously operated standards, the excess noise thus introduced is servoed out in the long term to become eventually smaller than the atomic shot noise, whereas in standards with pulsed operation the phase noise around even harmonics of the pulse rate is down-converted by aliasing to base band. This latter mechanism is referred to in the literature as Dick effect.

In this paper, a model of the frequency control servo system is proposed, in which the input signal is the (known) local oscillator (LO) phase noise $S_\phi(f)$ and the output signal is the (unknown) phase noise $S'_\phi(f)$ of the standard in closed loop operation. The level of excess white frequency noise added by aliasing on the stabilized LO through the Dick effect can be related analytically to the characteristics of the free LO phase noise. From this, the stability limitation (with slope $\tau^{-1/2}$) typical of the Dick effect can then be obtained by the usual conversion formulas based on the power law model.

I. INTRODUCTION

THE MECHANISM by which phase/frequency instabilities of the rf source induce an excess white frequency noise in pulse-operated Ramsey type atomic frequency standards has been recently given much attention and referred to as Dick effect, after it was first pointed out by Dick [1] and later studied more in depth in [2], [3]. This name is accepted here, as it seems appropriate and conveniently short.

In this paper it is shown that the Dick effect is introduced by the sampling of the rf phase that occurs in these devices. A contribution to clarity is offered as to how much of the Dick effect is due to the physics of the interaction between atoms and rf field, and how much can be thoroughly explained as a simple aliasing phenomenon in a sampling process. In fact, atomic physics is needed to analyze the details of how the phase is sampled within each interaction of the atoms with the rf field, particularly when the Rabi to Ramsey interaction time ratio is not negligible or the standard is operated at a power level other than optimum [4]. Such an analysis can be expected to improve the accuracy in the evaluation of the Dick effect, but the effect

itself is brought in by the phase sampling process and can be described with existing signal theory tools.

In order to analyze these effects, a model of the frequency control servo system is proposed, in which the input signal is the (known) LO phase noise $S_\phi(f)$ and the output signal is the (unknown) phase noise $S'_\phi(f)$ of the standard in closed loop operation. This approach makes the substance of the Dick effect extremely easy to grasp. Approximations guided by numerical simulation are used in the analysis in order to simplify results. With the help of these, simple formulas can be written, valid in most practical cases, by which the size of the effect can be evaluated with easy back-of-an-envelope calculations.

In the process of doing so, a short discussion is also offered in this paper on the meaning of the Power spectral densities in the case of nonstationary processes. Because phase noise is, in most cases, nonstationary in high quality frequency sources, this discussion is, in principle, necessary when spectral modifications are involved. In fact, in these cases, the power spectrum cannot be obtained from the Fourier transform of the autocorrelation.

II. PHYSICAL MODEL

In order to analyze the effects of LO phase noise on the stability of Ramsey type atomic frequency standards, the simplified Ramsey formula used in this paper is:

$$\text{signal} \propto \frac{1}{2}(1 + \cos(\phi_2 - \phi_1)). \quad (1)$$

In cold-atoms frequency standards (e.g., trapped ions or fountains), because of the narrow velocity distribution and the usually small Rabi to Ramsey interaction time ratio, this is a particularly good approximation of the actual transition probability, at least for the central Ramsey fringe. Appearing in (1) is the difference $\phi_2 - \phi_1$ between the phases of the microwave field as experienced by the atoms in the second and in the first interaction.

When using square wave frequency modulation of depth $\pm\Delta\nu$ around a frequency offset by an error $\delta\nu$ from the atomic resonance, the latter can be written as:

$$\phi_2 - \phi_1 = \pm 2\pi\Delta\nu T + 2\pi\delta\nu T + \varphi(t) - \varphi(t - T) \quad (2)$$

for either half of the modulation cycle. This is an acceptable approximation when the power is about optimum and the Rabi interaction time T_p is much shorter than

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with reference to Fig. 1. As already pointed out, functional blocks of Fig. 1 do not necessarily represent identifiable physical subsystems, but rather equations between signals.

Block H_1 is an LTI (linear time invariant) system representing (5), which can be represented in the frequency domain by means of its transfer function $|H_1(f)|e^{j\psi}$, with

$$|H_1(f)| = 2\sqrt{1 - \cos(2\pi fT)}\sqrt{1 + \cos(2\pi fT_c)} \quad (6)$$

and $\psi = \pi/2 - \pi f(T + T_c)$.

The relationship between the output signal $V_\varepsilon^p(t)$ and the input $V_\alpha(t)$ of the sample-and-hold (S/H) circuit can be written as:

$$V_\varepsilon^p(t) = \left[V_\alpha(t) \sum_{l=-\infty}^{+\infty} \delta(t - lT_s) \right] * h(t) \quad (7)$$

where $\delta(t)$ is the continuous delta function and $h(t)$ is the impulse response of an LTI system representing the hold operation of the SH circuit. The symbol $*$ stands for convolution. In our application $h(t)$ is a rectangular impulse equal to 1 for $t \in [0, T_s]$. By performing the convolution, (7) becomes $V_\varepsilon^p(t) = \sum_{l=-\infty}^{+\infty} V_\alpha(lT_s)h(t - lT_s)$. The sampling function ($V_\alpha(lT_s)$) and the holding function ($h(t - lT_s)$), which contribute to the process $V_\varepsilon^p(t)$, can be clearly recognized in (7).

The SH block is linear but is not time-invariant; therefore, it cannot be represented in the frequency domain by means of a transfer function. However, Fourier analysis is still feasible, as shown in Section IV.

Block H_2 is the loop integrator. Its transfer function is $H_2(f) = \frac{c}{jf}$.

IV. CLOSED LOOP PHASE NOISE

The scheme of Fig. 1 allows closed loop phase noise analysis by common signal theory techniques. However, it is important to point out that classical spectral analysis, developed for stationary processes, cannot be used. In fact, the signals of this loop cannot be modeled by stationary processes. The main reason for this is the presence of the SH block, which produces a nonstationary process at its output, even when the signal at the input is stationary¹.

Because of this, the different approach illustrated in Appendix A will be followed, and the spectral analysis of phase noise will not be done by the Fourier transform of the autocorrelation, but rather by operating on deterministic signals. In fact, the proposed model easily can be analyzed in the frequency domain when signals are deterministic.

In this way, it is possible to show that the relationship between the Fourier transforms of $\varphi(t)$ and $\varphi'(t)$, which

¹It is known from signal theory [9] that an SH block turns a stationary process into a cyclo-stationary process. A suitable random sampling operation can be introduced to overcome this problem, but this method is not applicable if the SH is inserted in a closed loop. The proof of this is somewhat involved, and will not be reported here because it is beyond the scope of this paper.

will be indicated by $\Phi(f)$ and $\Phi'(f)$, respectively, is²:

$$\Phi'(f) = \Phi(f) + H_e(f)\Gamma(f) \quad (8)$$

Here $H_e(f)$ represents the linear part of the loop gain, with the exception of $H_1(f)$, and includes the loop integrator $H_2(f)$, the VCO transfer function $\frac{A}{jf}$, and the holding function of the SH, while $\Gamma(f)$ is a periodic function of f , with period $f_s = 1/T_s$, given by:

$$\Gamma(f) = \frac{\sum_{l=-\infty}^{+\infty} \Phi(f - lf_s)H_1(f - lf_s)}{1 - \sum_{l=-\infty}^{+\infty} H_e(f - lf_s)H_1(f - lf_s)} \quad (9)$$

which contains the effect of aliasing. These equations make it possible to find the Fourier transform of $\varphi'(t)$ from the Fourier transform of $\varphi(t)$.

In order to obtain the power spectrum $S_{\varphi'}(f)$ of the random process $\varphi'(t)$ from the Fourier transform of a “deterministic” $\varphi'(t)$, which is nothing else than one of its realizations, the average indicated in (22) in Appendix A must be performed.

V. THE DICK EFFECT

Because the Dick effect intervenes at very low Fourier frequencies (well below the loop attack frequency), an approximation valid for $f \ll f_s$ can be used in analyzing the meaning of (8) and (9).

An obvious one easily can be identified for the denominator of (9), where the term for $l = 0$ in the sum is clearly much greater than all the other terms, as it contains the low frequency loop gain. By reducing the denominator to that term, and isolating the similar term in the sum at the numerator, (8) can be rewritten as

$$\Phi'(f) = -\frac{\sum_{l \neq 0} \Phi(f - lf_s)H_1(f - lf_s)}{H_1(f)} \quad (10)$$

The evaluation of the closed loop spectrum according to Appendix B involves the calculation of the expected value of the square modulus of (10), which can be written in a simple form only if the cross terms are considered negligible. Such an assumption is certainly acceptable if the phases of the (phase) noise around two different harmonics of f_s are uncorrelated, which seems realistic.

We can therefore write, for $f \ll f_s$,

$$S_{\varphi'}(f) \approx 2 \sum_{l=1}^{\infty} \frac{|H_1(lf_s)|^2}{|H_1(f)|^2} S_\phi(lf_s). \quad (11)$$

It is worth noting that this equation is similar in the form to the equation used in the literature to describe the Dick effect, with the “ g function” substituted by $H_1(f)$. The meaning of $|H_1(f)|$ here is rigorously defined by (6),

²Notice that the Fourier transform of a random process does not exist in general. Here $\varphi(t)$ and $\varphi'(t)$ are considered as deterministic signals (for example, each of them could be a single realization of the corresponding process in a finite time interval).

and for $f \ll f_s$ it can be seen to be $H_1(f) \approx 4\pi fT$. This latter approximation shows that the phase noise introduced by the Dick effect is white frequency noise, because $S_{\phi'}(f) \propto f^{-2}$. A simpler formula, still valid for any kind of phase noise process S_{ϕ} , can be derived from (11) by carrying also $H_1(lf_s)$ through a few easy passages, valid in the approximation $f \ll f_s$. It is then found that:

$$S_{\phi'}(f) \approx \frac{2}{f^2} \sum_{n=1}^{\infty} \left(\frac{\sin n\pi T/T_c}{n\pi T/T_c} \right)^2 (2nf_s)^2 S_{\phi}(2nf_s) \quad (12)$$

where the summing index l in (11) was substituted with $n = l/2$, because only terms with even l are left, as an odd l makes $H_1(lf_s)$ vanish. If expressed in terms of power spectral densities of fractional frequency fluctuations $S_y(f)$, (12) becomes even simpler:

$$S_{y'}(f) \approx 2 \sum_{n=1}^{\infty} \left(\frac{\sin n\pi T/T_c}{n\pi T/T_c} \right)^2 S_y(2nf_s). \quad (13)$$

Because the Dick effect instability contribution always turns out to be white frequency noise, the corresponding Allan variance is easily obtained from (13) by using the well-known formula

$$\sigma_{y'}^2(\tau) = \frac{S_{y'}(f)}{2\tau}. \quad (14)$$

For noise processes of the power law type, with $S_y(f) = h_{\alpha} f^{\alpha}$, (13) can be put in the form:

$$S_{y'}(f) = 2h_{\alpha} (2f_s)^{\alpha} F_{\alpha} \left(\frac{T}{T_c} \right) \quad (15)$$

where the function $F_{\alpha}(T/T_c)$ is given by:

$$F_{\alpha} \left(\frac{T}{T_c} \right) = \sum_{n=1}^{\infty} n^{\alpha} \left(\frac{\sin n\pi T/T_c}{n\pi T/T_c} \right)^2 \quad (16)$$

and summarizes the dependence of the Dick effect on the duty cycle T/T_c and on the spectral slope α . For $\alpha \geq 1$ the series in (16) diverges. However, an effective value of F_{α} still can be calculated for any given band limitation that may exist in the system, by truncating the series at the appropriate term. Curves of F_{α} versus T/T_c are shown in Fig. 2 for different values of α , including the first 30 (broken lines) or 3000 terms of the series (solid lines).

It is interesting to point out that, whatever the nature of the LO instability may be, the effect vanishes for $T = T_c$, which means that there is no dead time between interrogations, while at the limit for vanishing T is simply given by the infinite sum of the Power spectral densities of LO relative frequency fluctuations at the even harmonics of f_s :

$$S_{y'} \approx 2 \sum_{n=1}^{\infty} S_y(2nf_s). \quad (17)$$

In this latter case, band limitation is needed for (17) not to diverge for frequency noise processes with $\alpha \geq -1$, that

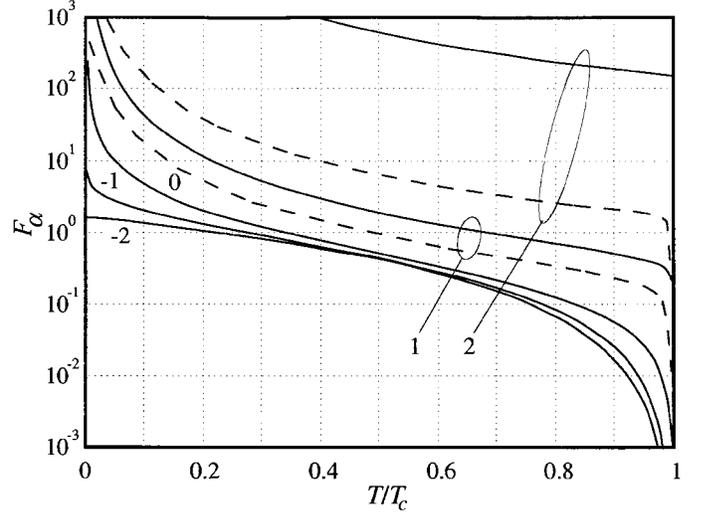


Fig. 2. Dependence of the Dick effect on the duty cycle T/T_c for different values of the spectral slope α . The only band limitation is given here by the finite number of terms summed up in the series (30 for the broken lines and 3000 for the solid lines). For $T_c = 1$ s this is equivalent to a bandwidth of 30 Hz and 3 kHz, respectively.

is not only for white frequency noise, but even for flicker of frequency. This is obviously an extreme case, unrealistic for frequency standards because T is the Ramsey drift time, which is always greater than zero in Ramsey schemes.

In between these extremes, the Dick effect assumes a finite value that depends on T/T_c and, for $\alpha \geq 1$, also on the bandwidth of the system. In Fig. 2 a band limitation is introduced by the finite number of terms summed up in the series (30 or 3000). For $T_c = 1$ s this is equivalent to a bandwidth of 30 Hz and 3 kHz, respectively. Incidentally, it may be worth pointing out here that the chosen 30 Hz bandwidth is compatible with existing devices [2], [4].

It also should be remembered at this point that block H_3 in Fig. 1 does exist in actual devices, though overlooked in this analysis, which is based on a simplified model. Block H_3 represents the effects of the phase averaging due to the finite Rabi interaction time T_p , which intrinsically introduces a band limitation. For details on the way this band limitation is introduced, quantum mechanical calculations are needed, which are beyond the scope of this paper and are treated in [4].

VI. LOOP STABILITY

For loop stability, what is really of interest here is that $S_{y'}(f)$ be smaller than $S_y(f)$ at low Fourier frequencies, because otherwise it would mean that the servo loop is actually worsening the stability of the LO beyond the attack time, which is not what one would expect of a stabilization servo. With this criterion, which can be written:

$$S_y(f) \gg 2 \sum_{n=1}^{\infty} \left(\frac{\sin n\pi T/T_c}{n\pi T/T_c} \right)^2 S_y(2nf_s), \quad (18)$$

conclusions can be made about the conditions of operation of the servo loop with different types of LO noise.

A general formula can be obtained, which is more stringent than (18), by substituting 1 for the sine function in all terms of the series, and supposing that the free LO frequency noise spectrum be a pure power law noise process. The condition is found:

$$S_y(f) \gg 2 \frac{2^\alpha S_y(f_s)}{(\pi T/T_c)^2} \zeta(2-\alpha) \quad (19)$$

where $\zeta(2-\alpha) = \sum_{n=1}^{\infty} n^{\alpha-2}$ is Riemann's Zeta function of the indicated integer argument [10]. This formula is valid for $\alpha \leq 0$. Because $S_y(f) = (f/f_s)^\alpha S_y(f_s)$, a condition can be found from (19) that does not contain the absolute level of frequency instability $S_y(f_s)$, but only its slope α . This is:

$$\left(\frac{T}{T_c}\right)^2 \gg 2 \frac{2^\alpha}{\pi^2} \zeta(2-\alpha) \left(\frac{f}{f_s}\right)^{-\alpha} \quad (20)$$

which shows that, for frequency noise in the LO with $\alpha < 0$, the loop is always operating correctly at sufficiently low Fourier frequencies, no matter what the duty cycle may be.

For $\alpha = 0$, or white frequency noise, it is $\zeta(2) = \pi^2/6$. The condition for useful loop operation (20) then becomes:

$$\frac{T}{T_c} \gg \frac{1}{\sqrt{3}} \approx 0.6. \quad (21)$$

This illustrates that it is not so much a low pass filter that is needed to ensure a properly operating loop in the presence of white frequency noise, as rather a duty cycle sufficiently close to 1.

VII. COMPARISON WITH NUMERICAL RESULTS

Numerical simulations of the loop of Fig. 1 in operation were made by using (8) and computing power spectra as spelled out in detail in Appendix B. The results of such simulations are reported in [11] and [12]. In particular in [11] results are given as a function of T/T_c for the Dick effect level in a system dominated by flicker of frequency noise.

The same numerical results are reported here for convenience in Fig. 3, where they are compared with the theoretical curve calculated from (14) and (15). For ease of comparison, the function F_α is used here instead of the parameter A of [11]. The relationship between the two is $F_\alpha(T/T_c) = A^2/2$.

VIII. CONCLUSIONS

In the analysis reported in this paper, a number of approximations were made. Although no deep discussions were carried out to justify them, they are strongly supported by the agreement between the derived formulas and the results of numerical simulations.

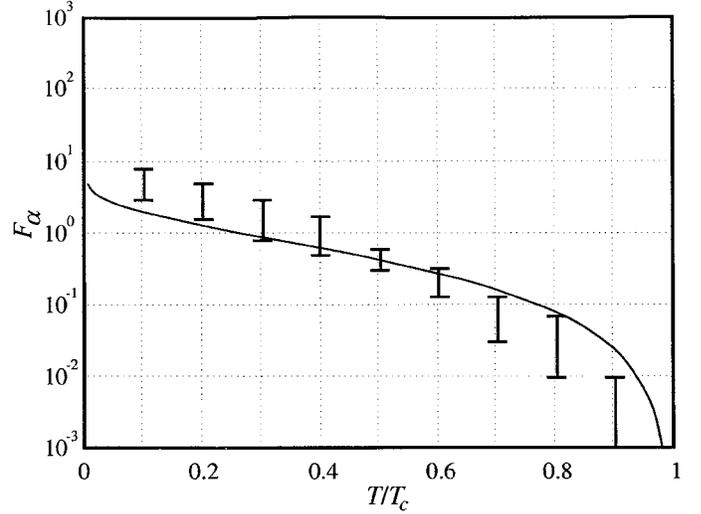


Fig. 3. Comparison of results obtained by numerical simulation of the loop operation with the theoretical curve obtained for the Dick effect as a function of the duty cycle T/T_c . The spectral slope for this run of data points was -1 (flicker of frequency noise).

The only possible parameters capable of affecting the level of the Dick effect appear to be the following:

- The level of phase/frequency stability of the local oscillator.
- The dead time between cycles, or T/T_c , as already pointed out in Section V.
- The sampling frequency (because of the increasing LO instabilities at low Fourier frequencies, a greater f_s means less noise).
- In some cases the band limitation (e.g., for white phase noise, or $\alpha = 2$).

Following the philosophy that the local oscillator cannot be readily improved, that is assuming that the best available oscillator is used and the problem still exists, only the handles offered by the last three parameters can be used to try and decrease the Dick effect.

For an analysis of band limitation in the white phase noise case, see [4]. As for acting on the other two parameters, a number of proposals exist.

In one proposal, two atomic pulsed operated resonators would be excited alternatively [2], in an effort to make the dead time vanish. In another one, the next shot would be prepared in a fountain while one is undergoing its parabolic flight [13] (the light shift problem would be taken care of, in this case, by blocking the light access to the drift region with a shutter that opens only to let the atoms through). In yet another proposal [14], there would be many shots in flight through the apparatus at all times.

Operation of a fountain with a continuous beam [5], [6] could, in principle, obliterate the Dick effect, though it may cause problems with the light shift. However, it must be pointed out that the modulation frequency of square wave frequency modulation must be very slow in this case, so that the blanking intervals that are necessary at the switching transients would not reintroduce aliasing and as

a consequence the Dick effect. As a result, the loop can start controlling only at extremely low frequencies (very long times), and the requirements of the LO are made more severe rather than relaxed, as oscillator instabilities tend to increase at lower Fourier frequencies. It is quite possible instead that this problem may be solved by the use of sinewave modulation, but in this case the intermodulation effect studied by Audoin et al. will show up [16].

APPENDIX A

FREQUENCY ANALYSIS OF A NONSTATIONARY PROCESS

The spectral analysis of a stationary process $X(t)$ is based on its Power spectrum $S_X(f)$, defined as the Fourier transform of its autocorrelation $R_X(\tau) = E\{X^*(t)X(t+\tau)\}$, where $E\{\cdot\}$ stands for expectation. If the process $X(t)$ is not stationary, the autocorrelation is a function of two variables (t , and τ), and this definition of power spectrum cannot be applied.

Several alternative definitions have been proposed [15] to introduce a function of frequency suitable to adequately describe a nonstationary process in the frequency domain. The method used here is based on the Power spectrum $\bar{S}_X(f)$, defined as the Fourier transform of the time average of the autocorrelation, that is the Fourier transform of:

$$\bar{R}_X(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_X(t+\tau, t) dt$$

where $R_X(t+\tau, t) = E\{X^*(t)X(t+\tau)\}$.

It can be shown [15] that $\bar{S}_X(f)$ also can be expressed as:

$$\bar{S}_X(f) = E \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \left| \int_{-T}^{+T} X(t) e^{-j2\pi ft} dt \right|^2 \right\}. \quad (22)$$

In our application this is a fundamental equation. In fact it is shown in Section IV that the Fourier transform contained in (22) (that is the integral $\int_{-T}^{+T} X(t) e^{-j2\pi ft} dt$) can be evaluated for both $\varphi(t)$ and $\varphi'(t)$, but the same is not possible for their power spectra.

Accordingly, power spectra used in the following are to be understood as defined like in (22). However, the simpler notation $S(f)$, without the overline, is always used in the text for simplicity.

APPENDIX B

ESTIMATE OF $S_{\varphi'}(f)$

In numerical simulations, the spectrum $S_{\varphi'}(f)$ is computed for all frequencies f_i of a regularly spaced grid, and each value $S_{\varphi'}(f_i)$ is obtained in three steps.

Step One

A number of realizations of the input phase noise are generated as numerical sequences by tayloring White Gaussian noise with numerical filters. This is done in a

way to ensure that their Fourier transform be consistent with the desired shape of the power spectrum $S_{\varphi}(f)$. Each realization is indexed by k , with $k = 1, 2, \dots, M$. For each frequency f_i and each realization k a complex spectrum sample $\Phi_k(f_i)$ is obtained.

Step Two

Equation (8) is used to compute a spectrum sample $\Phi'_k(f_i)$ for each frequency f_i and each realization.

Step Three

$\Phi'_k(f_i)$ is averaged over k , according to (22), yielding an estimate of $S_{\varphi'}(f_i)$.

It is important to point out that the described method is based on the existing analytical relationship between $\Phi(f_i)$ and $\Phi'(f_i)$. The simulation is used only to generate the spectrum samples $\Phi_k(f_i)$. Therefore, we can say that the nature of the phenomenon under examination is completely described by (8).

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